1 Introduction to Partial Differential Equations

We start with the natural question: what is a partial differential equation (PDE)?

Definition 1 (Partial Differential Equation)**.** *A PDE is an equation that relates a function to its partial derivatives.*

i.e.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t} - 1$$
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{1}$$

In order to be a PDE (vs. an ODE) the function described should have at *least* two different independent variables. i.e.

$$u(x,t) \quad \text{or} \quad u(x,y,t)$$

where the first had one spatial dimension and one temporal dimension and the second had two spatial dimensions and one temporal dimension. The convention in this course is that we will list spatial variables first and the temporal variable last.

Typically, we only consider one dependent variable (though exceptions will be seen) and *typically* we consider real valued u, x, t, etc. (i.e. unless otherwise specified assume u is of the form $u : \mathbb{R}^n \to \mathbb{R}$). While mathematically x, y, t, etc. are just arbitrarily labelled coordinates, we often talk about spatial and temporal variables differently.

1.1 Differential Operators

Any PDE can be written as

$$F(u, u_x, u_t, u_{xx}, u_{xt}, \ldots) = 0$$

i.e. for (1) we can write the PDE as

$$\frac{\partial}{\partial t}u - \frac{\partial^2}{\partial x^2}u = 0$$

Now, what is $\frac{\partial}{\partial t}$? i.e. what *kind* of mathematical object is it? It represents differentiation with respect to t. Hence, it is not a derivative itself, but rather represents the operation of taking a time derivative. This is but one example of a (differential) operator.

It's not the only kind of operator, consider the divergence and Laplacian operator from Calc 4 acting on a function u:

$$\nabla \cdot u = \sum_i \frac{\partial}{\partial x_i} u$$
$$\nabla^2 u = \sum_i \frac{\partial^2}{\partial x_i^2} u$$

In a similar vein if we define $L = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$, then (1) can be written as

$$\frac{\partial}{\partial t}u - \frac{\partial^2}{\partial x^2}u = 0$$
$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)[u] = 0$$
$$L[u] = 0.$$

Definition 2 (Linear Operator). *An operator $L : V \to W$ (where V and W are linear spaces) is said to be linear if*

1. $L[u_1 + u_2] = L[u_1] + L[u_2]$ *for all* $u_1, u_2 \in V$
2. $L[k\, u_1] = k\, L[u_1]$ *for all* $k \in \mathbb{R}$ *and* $u_1 \in V$

If L is a linear differential operator, then

1. $L[u] = f$ is a linear PDE
2. $L[u] = 0$ is a linear, homogeneous PDE

Theorem 1 (Principle of Superposition). *For a linear differential operator L if u_1 and u_2 are solutions of $L[u] = 0$ (i.e. $L[u_1] = L[u_2] = 0$), then $L[c_1\, u_1 + c_2\, u_2] = 0$ for all $c_1, c_2 \in \mathbb{R}$.*

Proof. Exercise (direct application of definition). □

So, we've made some definitions and invented notation: we know what a PDE is, can we solve any?

Example 1. *Solve the PDE*

$$\frac{\partial u}{\partial t} = -e^{-t}$$

for $u(x,t)$ such that $u(x,0) = (1+x^2)^{-1}$.

Solution:
Hold x fixed and integrate both sides with respect to t to see

$$\begin{aligned} \int \frac{\partial u}{\partial t}\,\mathrm{d}t &= \int -e^{-t}\,\mathrm{d}t \\ u(x,t) &= e^{-t} + C(x) \end{aligned}$$

where the constant of integration is replaced with an arbitrary function of x (as we're keeping x fixed). Applying the IC we have

$$u(x,0) = 1 + C(x) = \frac{1}{1+x^2} \implies C(x) = \frac{1}{1+x^2} - 1 = \frac{-x^2}{1+x^2}.$$

Hence

$$u(x,t) = e^{-t} - \frac{x^2}{1+x^2}$$

solves the PDE.