1 Equilibrium Solutions

We start with the natural question: what is a partial differential equation (PDE)?

Example 1. *Imagine a solid copper pipe used to move heat. Suppose one end is held at a constant temperature of* 0 *degrees Celsius and the other is held at a constant temperature of* b *degrees Celsius. The associated Initial Boundary Value Problem for the 1D heat equation is*

$$\begin{aligned} u(x,0) &= f(x) \\ u(0,t) &= 0 \\ u(L,t) &= 0 \\ \frac{\partial u}{\partial t} &= D\frac{\partial^2 u}{\partial x^2} \end{aligned} \tag{1}$$

What does the temperature of the pipe look like eventually?

In this example we will look at PDEs for which eventually the behaviour eventually stops changing in time. There are many PDEs with this property and we will look a bit at what conditions allow for this. For the example in equation (1), we assume that the following limit converges (as otherwise the question wouldn't make sense)
Let

$$U(x) = \lim_{t\to\infty} u(x,t).$$

Hence $\frac{\partial U}{\partial t} = 0$. But u solves (1), hence U also solves (1). Therefore,

$$D\,\frac{\partial^2 U}{\partial x^2} = 0$$

which is an ODE that can be readily solved to yield $U(x) = c_1\,x + c_0$. Now applying the BCs: $U(0) = c_0 = 0$ and $U(L) = c_1\,L + c_0 = b \implies U(x) = \frac{b}{L}\,x$. So for any initial temperature profile $f(x)$, the pipe will eventually cool down to an even gradient from b degrees at the right end to 0 degrees at the left.

Definition 1 (Steady State)**.** *For a general PDE if* $\lim_{t\to\infty} u(x,t)$ *converges, then* $u(x,t)$ *can be written as*

$$u(x,t) = v(x,t) + u_{eq}(x)$$

where $\lim_{t\to\infty} v(x,t) = 0$. *In this form* v *is called the transient state of the solution and* u_{eq} *is called the equilibrium state/steady state of the solution.*

It is not always the case that PDEs give rise to steady states. In infinite time solutions can do other things. For instance, the solution can diverge to infinity (either across the whole domain or across a subset); the solution could exhibit periodic behaviour (where it is always changing in time, but eventually settles down to a predictable cycle); or the solution could experience chaos (a very fun consequence of non-linearity and the study of future courses).
But when a system permits equilibrium states, they can be easier to solve for than the general solution. In chemical reactors equilibrium states often tell us the result of a chemical reaction (and are often assumed to happen quite quickly, hence the transient states are more safely ignored). In fact, equilibrium solutions of the heat equation are so important that the equation that defines them is one of our Big 3 Special Named PDEs of the course.

1.1 Laplace's Equation

$$\nabla^2 u(\vec{x},t) = 0$$

2 Derivation of the 1D Wave Equation

Constitutive relations and the conservation equation will yield many physical systems that are first order in time. For systems that are second order in time, one tool for finding equations is Newton's Second Law. We will use N2L to derive the third "famous" linear PDE in this course. This PDE is that which describes waves: either waves in strings (1D) or membranes (2^{+}D).
Consider a string stretched taut (i.e. a guitar string) with length l, density ρ, and tension τ. Let $y = u(x,t)$ represent the position of the string for $0 \le x \le l$. Then we make the following simplifying physical assumptions:

1. $u \ll l$ this implies that the string isn't stretched so far that the restoring tension causes a "string particle" to experience horizontal motion
2. $\left|\frac{\partial u}{\partial x}\right| \ll 1$ this (and the last one) say a similar thing. But here, we really are making the asymptotic simplification that $\left|\frac{\partial u}{\partial x}\right|^2 \approx 0$. This means that stretching of the string is negligible since
$$L = \int_0^l \sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2}\,\mathrm{d}x$$
3. The string is perfectly flexible (i.e. it does not resist bending, cf an iron bar or a coat hanger). This implies that the tension is tangent to the string (as non-tangent tensions would represnt internal forces that resist bending)

Now we focus on a string segment of length Δx at position x. The slope of the tangent line is given by $\frac{\partial u}{\partial x} = \tan(\theta)$ hence the y component of the tension at either end of the element is

$$\tau \frac{\partial u(x,t)}{\partial x} \quad \text{and} \quad \tau \frac{\partial u(x+\Delta x,t)}{\partial x}$$

Newton's second law says that the net force on the element is simply the mass of the element times the acceleration that the element experiences. Given our earlier assumptions, the only forces experienced is the difference in the vertical components of the tension at either end. Hence,

$$\begin{aligned} F_{\text{net}} &= m\,a \\ \tau \frac{\partial u(x+\Delta x,t)}{\partial x} - \tau \frac{\partial u(x,t)}{\partial x} &= (\rho\,\Delta x)\,\frac{\partial^2 u(x,t)}{\partial t^2} \\ \left(\frac{\tau}{\rho}\right) \frac{\frac{\partial u(x+\Delta x,t)}{\partial x} - \tau \frac{\partial u(x,t)}{\partial x}}{\Delta x} &= \frac{\partial^2 u(x,t)}{\partial t^2} \\ \implies \frac{\partial^2 u}{\partial t^2} &= (\tau/\rho)\,\frac{\partial^2 u}{\partial x^2} \end{aligned}$$

which yields our third famous PDE: the Wave Equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\,\nabla^2 u$$

where c is called the wave speed.