1 Hadamard's Notions of Well Posedness

We've seen various "realism" conditions via the conservation law and Newton's second law. However, there is another type of "realism" condition that is often regarded as essential in physical (biological, chemical, ...) systems. These form **Hadamard's Notions of Well Posedness** and are motivated by experience.
First the physical motivations

P_1: the physical system exists

P_2: the same experiment has the same outcome

P_3: if the conditions change a little, then the outcome only changes a little

and each corresponding mathematical condition

M_1: the math model has at least one solution ("existence")

M_2: the model as at most one solution ("uniqueness")

M_3: the solution must depend continuously on ICs and BCs. ("continuous dependence")

An IBVP for a PDE which satisfies M_1, M_2, and M_3 is called **well-posed**, otherwise it is called **ill-posed**.
Throughout this course we will focus on M_1 as we develop solution methods and will dabble in M_2 (energy methods, etc.). M_3 is left for further courses (AMATH 453 and others). M_3 is deeply related to questions of nonlinear dynamics and chaos and while that area of inquiry is incredibly rich, deep, and fun, the curious student should be forewarned: *hic sunt dracones.*

2 How *fast* is a PDE?

We saw last class that sometimes PDEs will eventually settle down to a solution that stops changing in time. It is natural to ask: how long until we get there? Most of the time the answer to that question is: ∞. Which is not particularly helpful. Realistically, if we were imagining a cup of coffee cooling down on a counter or a uranium rod cooling in a bath we know that *true* convergence takes an infinite amount of time, but practically my coffee is cold after 10 minutes. How long does it take for the uranium rod to cool? Maybe not so precisely, but is it on the order of minutes, hours, years?
Consider the PDE

$$L[u] = 0, \quad u(0,t) = u_0, u(l,t) = u_l, u(x,0) = f(x)$$

we will consider the question of time-scales under our three favourite forms of the linear differential operator.
Let $L = \partial_t - D\,\partial_x^2$ (the heat/diffusion equation). How do we combine our constants to get units of time? Well, $[D] = \text{length}^2/\text{time}$, $[l] = \text{length}$, hence if we take $T = l^2/D$ then T will have units of time. In this case we say T is our **time scale**. (Intuitively convince yourself that T should decrease as D increases and that T should increase as l increases, note that this intuitive argument might not be enough to show that T would depend on l quadratically).
For the wave equation if $L = \partial_t^2 - c^2\,\partial_x^2$, then what is our time scale? Well $[c] = \text{length}/\text{time}$ hence $T = l/c$ has units of time.
What about for Laplace's equation? What if $L = \nabla^2$? There is no natural time scale! Which should make sense, Laplace's equation is physically interpreted as the steady state of the heat equation. Steady states are constant in time and so have no time dependence.

All this discussion is well and good, but there's an elephant in the room: what is a time scale? Intuitively we define it as "the length of time for a solution to change significantly". Which is loose and hand-wavey on purpose.

Imagine a situation like $u(x,t) = e^{-k\,t}F(x)$ for some bounded function F. Such a function is exponentially decaying to 0 as time grows. In this case, a natural interpretation of a time scale is something like a half-life (the amount of time required for the initial concentration to be reduced by a factor of 2). What if instead $u(x,t) = \sin(\omega\,t)\,F(x)$ for a bounded F. Then a half life makes little sense in this context, but sin is periodic and so a period is a reasonable time scale to choose. Hence $T = 2\,\pi/\omega$ is a natural time scale for this solution. Note that if we were to use our dimensional analysis argument on the solution, we could deduce that ω has units $[\omega] = 1/\text{time}$ (this is because sin needs its argument to be dimensionless, remember sin can be thought of as a Taylor series and we can only add things of like units). From this dimensional analysis point of view, we'd only be able to deduce that $T = 1/\omega$. In particular, we're off by a factor of $2\,\pi$. Sometimes mathematicians call this "qualitatively" but not "quantitatively" correct. Meaning, in an order of magnitude sense it describes the correct dependence on our parameters (so qualitatively it describes the correct solution), but it is quantitatively incorrect because the actual number does not directly line up. We will circle back to this concept of time scales (and the related length scales) after we solve the heat and wave equations. In particular, diffusion has some really strange behaviour at differing length scales.

3 Solution Methods

Let's solve some PDEs.

Recall the example from the very first class with $u_t = -e^{-t}$: we were able to solve this "PDE" because the spatial variable does not explicitly appear in the equation. Hence, by holding x constant, we needed to only integrate.

Similarly, for the next example we will solve it by integrating each variable one at a time.

Example 1. *Solve*

$$u_{xt} - 3\,u_x = e^{-x}$$

for $u(x,t)$.

Rearranging the equation yields

$$\begin{aligned}
\frac{\partial}{\partial x}\left(u_t - 3\,u\right) &= e^{-x} \\
\int \frac{\partial}{\partial x}\left(u_t - 3\,u\right)\,\mathrm{d}x &= \int e^{-x}\,\mathrm{d}x \\
u_t - 3\,u &= -e^{-x} + f(t)
\end{aligned}$$

This is a first order linear ODE! Hence we solve via the **method of integrating factors**. To start consider the integrating factor $\mu(t) = e^{\int -3\,\mathrm{d}t} = e^{-3t}$. Multiplying both sides by $\mu(t)$ we get

$$\begin{aligned}
e^{-3t}\,u_t - e^{-3t}\,3\,u &= -e^{-3t}\,e^{-x} + e^{-3t}\,f(t) \\
\frac{\partial}{\partial t}\left(e^{-3\,t}\,u\right) &= -e^{-3t}\,e^{-x} + e^{-3t}\,f(t) \\
\int \frac{\partial}{\partial t}\left(e^{-3\,t}\,u\right)\mathrm{d}t &= \int -e^{-3t}\,e^{-x} + e^{-3t}\,f(t)\,\mathrm{d}t \\
u &= \frac{1}{3}e^{-x} + F(t) + g(x).
\end{aligned}$$

If we had initial conditions or boundary conditions we could apply them here to find the form of F and g. Solving PDEs seems easy! You just deal with one variable at a time. Unfortunately, that doesn't always work (consider the heat equation, try as you may there's no way to integrate time first and then space or vice-versa). Generally, we can't do this. But this what was nice about these two examples! We will start now with first order equations and see if we can come up with a general way to trade solving a PDE for solving multiple ODEs (or ODEs and integrations).

3.1 First Order Linear PDEs

The most general first order linear PDE in 1D is

$$a(x,t)\,\frac{\partial u}{\partial t} + b(x,t)\,\frac{\partial u}{\partial x} + c(x,t)\,u + d(x,t) = 0$$

First, if $a = 0$ or $b = 0$ then we already know how to solve this (it's just an ODE). So for now we assume that a and b are not identically zero.
We can't, in general, integrate one dependent variable at a time. So we want to seek a change of variables to build to that end. However we first need a technical lemma..

Lemma 1. *If $\psi(x,t) = k$ is the general (implicit) solution of $\frac{\mathrm{d}x}{\mathrm{d}t} = F(x,t)$ then $\psi(x,t)$ satisfies*

$$\frac{\psi_t}{\psi_x} = -F.$$