Lemma 1. *If $\psi(x,t) = k$ is the general (implicit) solution of $\frac{\mathrm{d}x}{\mathrm{d}t} = F(x,t)$ then $\psi(x,t)$ satisfies*

$$\frac{\psi_t}{\psi_x} = -F.$$

Proof. Write $\psi(x,t)$ as $\psi(x(t),t)$ and then,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\,\psi(x(t),t) &= \frac{\mathrm{d}}{\mathrm{d}t}k \\ \psi_x\,x' + \psi_t &= 0 \\ \psi_x\,F(x,t) + \psi_t &= 0 \\ \frac{\psi_t}{\psi_x} &= -F(x,t) \end{aligned}$$

□

With that out of the way, we're ready to see our general method for solving first order linear PDEs (and we'll later use this same method for some non-linear PDEs and second order equations).

1 The Method of Characteristics

We change variables

$$\xi = \xi(x,t) \quad \eta = \eta(x,t)$$

In order to be invertible, we recquire the Jacobian to be non-singular. That is

$$\det\left(J(\xi,\eta)\right) = \begin{vmatrix} \xi_x & \xi_t \\ \eta_x & \eta_t \end{vmatrix} = \xi_x\,\eta_t - \xi_t\,\eta_x \neq 0$$

Since the change of variables is invertible, we can think of x and t as being functions themselves

$$x = x(\xi,\eta), \quad t = t(\xi,\eta)$$

Notationally,

$$\hat{u}(\xi,\eta) = u(x(\xi,\eta), y(\xi,\eta))$$

That is, when we're in translated coordinates we write $\hat{u}$ and in physical coordinates we write x. We can do this because we know that the change of variables is invertible (and so no information is lost when going from u to $\hat{u}$).
Calculating derivatives in terms of new variables and we see

$$\frac{\partial u}{\partial x} = \frac{\partial \hat{u}}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \hat{u}}{\partial \eta}\frac{\partial \eta}{\partial x}$$

and, similarly,

$$u_t = \hat{u}_\xi\,\xi_t + \hat{u}_\eta\,\eta_t$$

so the PDE

$$a(x,t)\,u_t(x,t) + b(x,t)\,u_x(x,t) + c(x,t)\,u(x,t) + d(x,t) = 0$$

becomes

$$a\,\left(\hat{u}_\xi\,\xi_t + \hat{u}_\eta\,\eta_t\right) + b\,\left(\hat{u}_\xi\,\xi_x + \hat{u}_\eta\,\eta_x\right) + c\,\hat{u} + d = 0$$

or, rearranging,

$$\left(a\,\xi_t + b\,\xi_x\right)\hat{u}_\xi + \left(a\,\eta_t + b\,\eta_x\right)\hat{u}_\eta + c\,\hat{u} + d = 0.$$

This doesn't look any easier, but let's choose η such that $a\,\eta_t + b\,\eta_x = 0$. Or, rearranging,

$$\frac{\eta_t}{\eta_x} = -\frac{b}{a}.$$

Hence to find such an η we need to use our lemma. That is, $\eta(x,t) = \psi(x,t)$ where $\psi(x,t)$ is the general (implicit) solution of the Characteristic ODE

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{b(x,t)}{a(x,t)}$$

After which the original PDE reduces to a first order, linear ODE

$$(a\,\xi_t + b\,\xi_x)\,\hat{u}_\xi + c\,\hat{u} + d = 0$$

Now, for ξ we only require that

$$\xi_x\,\eta_t - \xi_t\,\eta_x \neq 0$$

hence $\xi = t$ satisfies. After which the ODE reduces to

$$a(x(\xi,\eta),t(\xi,\eta))\,\hat{u}_\xi + c(x(\xi,\eta),t(\xi,\eta))\,\hat{u} + d(x(\xi,\eta),t(\xi,\eta)) = 0$$

or

$$\hat{a}\,\hat{u}_\xi + \hat{c}\,\hat{u} + \hat{d} = 0$$

Note: Could also have taken $\xi = \psi(x,t)$ and $\eta = x$ to eliminate the $\hat{u}_\xi$ term. Sometimes one is easier than the other.

Example 1. *The linear-advection equation (Lec 4) is given by*

$$\frac{\partial u}{\partial t} + k\,\frac{\partial u}{\partial x} = 0$$

with initial data $u(x,0) = e^{-x^2}$ *and* $-\infty < x < \infty$ *(and hence, no boundary conditions needed). Find the solution using the method of characteristics.*

In this example $a(x,t) = 1$, $b(x,t) = k$, and $c(x,t) = d(x,t) = 0$. Then, the characteristic ODE is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{b(x,t)}{a(x,t)} = k$$

Hence $x(t) = k\,t + c_0$ or $x - k\,t = c_0$ and so $\psi(x,t) = x - k\,t$
(Aside: as a sanity check note that $\psi_t = -k$ and $\psi_x = 1$, hence $a\,\eta_t + b\,\eta_x = (1)\,(-k) + (k)\,1 = 0$. Which is what we wanted: we wanted the change of variables to cancel it out.)
Let $\eta = x - k\,t$ with $\xi = t$, then the PDE becomes

$$\begin{aligned}(a\,\xi_t + b\,\xi_x)\,\hat{u}_\xi &= 0\\ \hat{u}_\xi &= 0\\ \implies \hat{u} &= f(\eta)\end{aligned}$$

Hence

$$\hat{u}(\xi,\eta) = f(\eta) \implies u(x,t) = f(x - k\,t)$$

and so

$$u(x,0) = f(x) = e^{-x^2} \implies u(x,t) = f(x - k\,t) = e^{-(x-k\,t)^2}.$$

Just as an aside in this problem $\eta(x,t) = x - k\,t$ and $\xi(x,t) = t$ hence $x(\xi,\eta) = \eta + k\,\xi$ and $t(\xi,\eta) = \xi$.

1.1 Summary

To solve

$$a(x,t)\,u_t + b(x,t)\,u_x + c(x,t)\,u + d(x,t) = 0 \tag{1}$$

Take

$$\xi = t, \quad \eta = \psi(x,t)$$

where $\psi(x,t) = k$ is the general (implicit) solution of the **characteristic ODE**

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{b(x,t)}{a(x,t)}$$

The PDE (1) then becomes

$$\hat{a}\,\hat{u}_\xi + \hat{c}\,\hat{u} + \hat{d} = 0 \tag{2}$$

where $\hat{a} = a(x(\xi,\eta), t(\xi,\eta))$, etc. Equation (2) is a first order, linear ODE and can be solved via the method of integrating factors.