1 Method of Characteristics

1.1 First order PDEs

Example 1. *Solve the BVP*

$$x\,u_x + u_t + u = 0, \quad u(1,t) = t$$

To start we consider the characteristic ODE

$$x'(t) = x \implies x = A\,e^t \implies \ln(x) - t = B$$

therefore $\psi(x,t) = \ln(x) - t$. In this example, I will let $\xi = \psi(x,t)$ and $\eta = x$ (I do this because I see an x on its own in the PDE and this makes the algebra easier).
So

$$u_x = \hat{u}_\xi\,\xi_x + \hat{u}_\eta\,\eta_x = \frac{1}{x}\hat{u}_\xi + \hat{u}_\eta, \quad u_t = \hat{u}_\xi\,\xi_t + \hat{u}_\eta\,\eta_t = -\hat{u}_\xi$$

Hence the PDE reduces to

$$0 = \eta\left(\frac{1}{\eta}\hat{u}_\xi + \hat{u}_\eta\right) + (-\hat{u}_\xi) + \hat{u} = \eta\,\hat{u}_\eta + \hat{u}$$

which can immediately be solved for

$$\hat{u} = \frac{f(\xi)}{\eta}$$

hence, in physical variables,

$$u(x,t) = \frac{f(\ln(x) - t)}{x}.$$

To find f take $u(1,t) = t$ to see

$$f(-t) = t \implies f(s) = -s$$

thus

$$u(x,t) = \frac{t - \ln(x)}{x}.$$

1.2 Second order PDEs

Earlier we saw that the linear differential operator used in the wave equation could be factored so that the wave equation could be rewritten. i.e:

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0 \implies \left(\frac{\partial}{\partial t} + c\,\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\,\frac{\partial}{\partial x}\right) u = 0$$

or, if $L_\pm = \partial_t \pm c\,\partial_x$ then the wave equation becomes

$$L_+[L_-[u]] = 0.$$

We then noted that we could start with $L_-[u] = 0$ or $L_+[u] = 0$ and either option would work. Which while true, does not let us close the system. However, at the time we didn't know how to solve first order linear PDEs and now we do (and so we can do better than this).

Example 2. *Solve the system $L_+[L_-[u]] = 0$ on the domain $-\infty < x < \infty$, $t \geq 0$ subject to the initial conditions*

$$u(x,0) = u_0(x), \quad u_t(x,0) = v_0(x)$$

To do so we will solve two systems. First we'll find a solution v such that

$$L_+[v] = 0$$

then we'll find u such that

$$L_-[u] = v$$

which will solve this decoupled system of two linear first order PDEs.
To start, last class we saw that $L_+[v] = 0$ has the general solution

$$v(x,t) = f(x - c\,t)$$

where f is some arbitrary function.
Now, to solve

$$L_-[u] = v \implies \frac{\partial u}{\partial t} - c\,\frac{\partial u}{\partial x} = f(x - c\,t).$$

We first consider the characteristic ODE

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -c \implies \psi(x,t) = x + c\,t$$

then we let $\eta = \psi(x,t) = x + c\,t$ and $\xi = t$ after which the PDE becomes

$$\hat{u}_\xi = f(x - c\,t)$$

Now this is not closed, as I have to write the RHS in terms of ξ and η.

$$\hat{u}_\xi = f(\eta - 2\,c\,\xi)$$

Therefore, we find

$$\begin{aligned} \hat{u} &= \int f(\eta - 2\,c\,\xi)\,\mathrm{d}\xi + G(\eta) \\ &= -\frac{1}{2\,c}\int f(s)\,\mathrm{d}s + G(\eta) \\ &= F(\eta - 2\,c\,\xi) + G(\eta) \end{aligned}$$

where $F(s) = -\frac{1}{2c}\int f(s)\,\mathrm{d}s$.
Or, in terms of our physical variables,

$$u(x,t) = F(x - c\,t) + G(x + c\,t) \tag{1}$$

which is the general form of the solution of the wave equation.
We can use our initial data to find the form of F and G in Equation (1). That is,

$$\begin{aligned} u(x,0) &= u_0(x) = F(x) + G(x) \\ u_t(x,0) &= v_0(x) = -c\,F'(x) + c\,G'(x) \end{aligned}$$

From which we can determine F and G. For instance,

$$\begin{aligned} F(x) &= u_0(x) - G(x) \\ \implies F'(x) &= u_0'(x) - G'(x) \\ \implies v_0(x) &= -c\,(u_0'(x) - G'(x)) + c\,G'(x) \\ \implies 2c\,G'(x) &= v_0(x) + c\,u_0'(x) \\ \implies G'(x) &= \frac{1}{2c}v_0(x) + \frac{1}{2}\,u_0'(x) \end{aligned}$$