1 Previously...

We saw that the general solution to the wave equation was

$$u(x,t) = F(x - c\,t) + G(x + c\,t)$$

and now applying the initial conditions

$$\begin{aligned} u(x,0) &= u_0(x) = F(x) + G(x) \\ u_t(x,0) &= v_0(x) = -c\,F'(x) + c\,G'(x) \end{aligned}$$

from which we can determine F and G. For instance,

$$\begin{aligned} F(x) &= u_0(x) - G(x) \\ \implies F'(x) &= u_0'(x) - G'(x) \\ \implies v_0(x) &= -c\,(u_0'(x) - G'(x)) + c\,G'(x) \\ \implies 2c\,G'(x) &= v_0(x) + c\,u_0'(x) \\ \implies G'(x) &= \frac{1}{2c} v_0(x) + \frac{1}{2}\,u_0'(x) \end{aligned}$$

2 Continuing d'Alembert's Solution

I'm thinking ahead here and recognizing that I want to evaluate G at the point $x + c\,t$ and so I want to write my solution in a way that lends itself easier to substitution, so I'll make use of the fact that by FTOC

$$\int h(x)\,\mathrm{d}x + C = \int_a^x h(s)\,\mathrm{d}s + \tilde{C}$$

where a is any point in the domain of h.
Thus,

$$G(x) = \frac{1}{2c}\int_a^x v_0(s)\,\mathrm{d}s + \frac{1}{2}u_0(x) + C_1$$

and so

$$F(x) = u_0(x) - G(x) = \frac{1}{2}\,u_0(x) - \frac{1}{2c}\int_a^x v_0(s)\,\mathrm{d}s - C_1.$$

Subbing these into our general solution, we see

$$\begin{aligned} u(x,t) &= F(x - c\,t) + G(x + c\,t) \\ &= \frac{1}{2}\,u_0(x - c\,t) - \frac{1}{2c}\int_a^{x-c\,t} v_0(s)\,\mathrm{d}s - C_1 + \frac{1}{2c}\int_a^{x+c\,t} v_0(s)\,\mathrm{d}s + \frac{1}{2}u_0(x + c\,t) + C_1 \\ &= \frac{u_0(x - c\,t) + u_0(x + c\,t)}{2} + \frac{1}{2c}\int_{x-c\,t}^{a} v_0(s)\,\mathrm{d}s + \frac{1}{2c}\int_a^{x+c\,t} v_0(s)\,\mathrm{d}s \\ &= \frac{u_0(x - c\,t) + u_0(x + c\,t)}{2} + \frac{1}{2c}\int_{x-c\,t}^{x+c\,t} v_0(s)\,\mathrm{d}s \end{aligned}$$

which is called *d'Alembert's solution* after the French mathematician Jean-Baptiste d'Alembert.

3 Classification of Second Order PDEs

In the previous section we saw that factoring the differential operator was a powerful trick. Hence the question of whether or not we'll be able to factor the operator into a product of two first order operators is important.

1. Wave Equation: Can factor into two distinct operators
2. Heat Equation: Cannot factor $\partial_t - D\,\partial_x^2$ as two linear, first order operators. If we *only* factor the 2nd order part, then we can factor it with multiplicity two as $\partial_t - (\sqrt{D}\partial_x)^2$
3. Laplace's Equation (in 2D): could factor using complex numbers $\partial_x^2 + \partial_y^2 = (\partial_x + i\,\partial_y)\,(\partial_x - i\,\partial_y)$, but this isn't *particularly* helpful

These three PDEs are qualitatively different with respect to the way the linear operator factors. As a consequence of the fundamental theorem of algebra, these are the only options for factoring the higher order terms of a second order linear operator: either two unique (real valued) factors, two factors that form a complex conjugate pair, or a repeated double (real valued) factor. It is for this reason that we consider the wave equation, heat equation, and Laplace's equation as prototypical examples of linear second order PDEs. For these three prototypical equations it is straightforward to factor the operator as the operator is relatively simple (and the coefficients are constant), in general it is not so easy to factor an operator and so we wish to develop a more general test by which we can determine if an operator is factor-able. It is that goal that we now turn ourselves towards.
Consider the most general second order linear PDE:

$$A(x,t)\,\frac{\partial^2 u}{\partial t^2} + 2\,B(x,t)\,\frac{\partial^2 u}{\partial x\,\partial t} + C(x,t)\,\frac{\partial^2 u}{\partial x^2} + D(x,t)\,\frac{\partial u}{\partial x} + E(x,t)\,\frac{\partial u}{\partial t} + F(x,t)\,u = G(x,t) \tag{1}$$

(Now you might be surprised by the factor of 2 in front of the B term, including this factor of 2 is the convention. Including the 2 here saves us from having to write factors of 1/2 later. Certainly no generality is lost by this convention.)
We want to factor the higher order terms of the linear operator and so, for the time being, we concern ourselves only with the highest ordered terms and desire to factor

$$A(x,t)\,\frac{\partial^2 u}{\partial t^2} + 2\,B(x,t)\,\frac{\partial^2 u}{\partial x\,\partial t} + C(x,t)\,\frac{\partial^2 u}{\partial x^2} + \text{lower ordered terms} = 0$$

To clean this up, we divide through by $A(x,t)$ to find

$$\frac{\partial^2 u}{\partial t^2} + 2\,\frac{B}{A}\,\frac{\partial^2 u}{\partial x\,\partial t} + \frac{C}{A}\,\frac{\partial^2 u}{\partial x^2} + \text{lower ordered terms} = 0 \tag{2}$$

where arguments are suppressed for ease of notation.
Now I *want* to factor this into something that looks like

$$\left(\frac{\partial}{\partial t} - \omega^-(x,t)\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - \omega^+(x,t)\frac{\partial}{\partial x}\right) u + \text{lower ordered terms} = 0 \tag{3}$$

At this point it is not *obvious* why I would call the coefficients of my factors $\omega^\pm$, but that will become obvious in a second. For now, just think of these two functions as arbitrary functions.
Let's find the form of $\omega^\pm$ by expanding out (3) and comparing it with (2).

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \omega^-\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - \omega^+\frac{\partial}{\partial x}\right) u &= \left(\frac{\partial}{\partial t} - \omega^-\frac{\partial}{\partial x}\right)\left(\frac{\partial u}{\partial t} - \omega^+\frac{\partial u}{\partial x}\right)\\
&= \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t} - \omega^+\frac{\partial u}{\partial x}\right) - \omega^-\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} - \omega^+\frac{\partial u}{\partial x}\right)\\
&= \frac{\partial^2 u}{\partial t^2} - \frac{\partial \omega^+}{\partial t}\frac{\partial u}{\partial x} - \omega^+\frac{\partial^2 u}{\partial t\,\partial x} - \omega^-\frac{\partial^2 u}{\partial x\,\partial t} + \omega^-\frac{\partial \omega^+}{\partial x}\frac{\partial u}{\partial x} + \omega^-\omega^+\frac{\partial^2 u}{\partial x^2}
\end{aligned}$$

If I compare with (2) I see that

$$2\,\frac{B}{A} = -(\omega^+ + \omega^-), \quad \frac{C}{A} = \omega^-\,\omega^+$$

it is straightforward to solve these equations simultaneously to receive

$$\omega^\pm = \frac{-B \pm \sqrt{B^2 - AC}}{A}.$$

At which point we notice two things: first we were vindicated in our decision to labelling these as $\omega^\pm$ and second we have found the "easy" condition by which we can determine if the linear operator factors. That condition is based on the sign of

$$B^2 - AC.$$

We are now ready for the following definitions:

Definition 1 (Discriminant)**.** *The **discriminant** of the general second order linear PDE* (1) *is given by* $B^2 - AC$*.*

Definition 2 (2nd Order PDE Class)**.** *For the second order linear PDE* (1)

- *If* $B^2 - AC > 0$*, then the PDE is called **hyperbolic***
- *If* $B^2 - AC = 0$*, then the PDE is called **parabolic***
- *If* $B^2 - AC < 0$*, then the PDE is called **elliptic***