1 Classification of Second order PDEs

If we think in terms of our three prototypical equations, then the heat equation with diffusivity of 1 corresponds to $A = 0$, $B = 0$, $C = 1$, $D = 0$, $E = -1$, $F = 0$, and $G = 0$ and hence has a discriminant of zero. Thus the heat equation is a parabolic equation. The wave equation with wave-speed of 1 has $A = 1$, $B = 0$, $C = -1$, $D = 0$, $E = 0$, $F = 0$, and $G = 0$ and so has a discriminant of 1. Therefore the wave equation is a hyperbolic equation. For Laplace's equation, take $t = y$ (since Laplace's equation is thought of as being independent of time) in which case $A = 1$, $B = 0$, $C = 1$, $D = 0$, $E = 0$, $F = 0$, and $G = 0$ and so has a discriminant of -1. Therefore Laplace's equation is elliptic.
Note that for these prototypical examples, the coefficient functions were constants. In general, coefficient functions can vary as functions of space and time. Therefore, the class of the PDE can vary dependent upon position in the xt plane. For instance,

$$\frac{\partial^2 u}{\partial t^2} = x \frac{\partial^2 u}{\partial x^2}$$

has discriminant x and so is hyperbolic when $x > 0$, elliptic when $x < 0$, and parabolic when $x = 0$.

2 Normal Mode Solutions: Stability and Dispersion

A benefit of classifying PDEs is that we can relate more complicated PDEs to their more simple class-mates. For instance, the wave equation is but one hyperbolic PDE but it's reasonable to assume that *other* hyperbolic PDEs have wave-like solutions.
We will now see that wave-like solutions show up in a lot more than just hyperbolic equations, in fact wave-like solutions are common solutions of second order constant coefficient PDEs.

Definition 1 (Normal Mode Solution)**.** *A normal mode solution of a PDE is a solution that is a (superposition) of sinusoids.*

For our purposes we often think of normal mode solutions as those that look like

$$u(x,t) = A(k)\,\exp\left(i\,k\,x + \lambda(k)\,t\right)$$

for particular $A(k)$, k, and $\lambda(k)$ values. In this notation we call k the *wavenumber* with associated wavelength $2\pi/k$ and $A(k)$ the amplitude. The form of λ is often determined by the particular PDE being solved.
(As an aside: the importance of Fourier analysis cannot be over-stated here: the (rather surprising) result that any well-behaved function can be written as the (infinite) sum of sinusoids is arguably one of the most ubiquitous results of numerical calculus in the natural sciences and engineering. It is precisely because of this convenient result (and the superposition principle for linear PDEs) that normal mode solutions are such an interesting thing to study.)

Example 1. *For $A = 1$, find the normal mode solutions of the wave equation.*

Taking

$$u(x,t) = \exp\left(i\,k\,x + \lambda(k)\,t\right),$$

then it is the case that

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left(\exp\left(i\,k\,x + \lambda(k)\,t\right)\right) = \lambda(k)\,\exp\left(i\,k\,x + \lambda(k)\,t\right) = \lambda(k)\,u$$

similarly

$$\frac{\partial^2 u}{\partial t^2} = \lambda^2\,u, \quad \frac{\partial u}{\partial x} = i\,k\,u, \quad \frac{\partial^2 u}{\partial x^2} = -k^2\,u.$$

Hence, when substituting into the wave equation, we see

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} u$$
$$\lambda^2(k)\, u(x,t) = -k^2\, c^2\, u(x,t)$$

Thus either $u = 0$ or

$$\lambda^2 = -k^2\, c^2 \implies \lambda(k) = \pm\, i\, k\, c$$

so, for any value of k, we have that

$$u(x,t) = \exp\left(i\, k\, (x \pm c\, t)\right)$$

solves the wave equation. This matches what we saw earlier for the general form of the wave equation: all solutions will be functions of just $x \pm c\, t$ (here we're just seeing a *particular* instantiation of this). Moreover, this tells us that if a solution of this PDE ever takes on the form of a wave, it will stay as a wave for all time.

But we can apply this to non-wave equations too, for instance in the context of the diffusion equation using the same normal-mode solution we see

$$\partial_t\, u = D\, \partial_{xx}\, u$$
$$\lambda\, u = -D\, k^2\, u$$
$$\implies \lambda(k) = -D\, k^2$$

hence,

$$u(x,t) = \exp(i\, k\, x - D\, k^2\, t)$$

Thus any waves that may occur in the diffusion equation will be exponentially smoothed out in time.

2.1 Stability

For Laplace's equation we see something different,

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} = 0$$

If we take our normal mode solution here, we see

$$\lambda^2\, u - k^2\, u = 0 \implies \lambda(k) = \pm k.$$

Let's now consider our form of our solution for different values of (k, λ). If $(k, \lambda) = (1, -1)$, then our normal mode solution is

$$u(x,t) = \exp[i\, x - t]$$

notably a sinusoid that is decaying in time. Instead, if $(k, \lambda) = (1, 1)$, then our normal mode solution is

$$u(x,t) = \exp[i\, x + t]$$

a sinusoid that is exponentially exploding in time. This is an undesirably trait in many physical PDEs. For instance, exponentially growing waves can be a result of resonance and great failure in physical systems can be achieved when u grows too large in finite time (i.e. mathematically $u \to \infty$ as $t \to \infty$, in practice u getting above some threshold in finite time is enough cause for concern).

Another example is given by the Helmholtz equation (Ex. 2.4.2, p. 34 Poulin)

$$u_{xx} + u_{tt} + \rho\, u = 0$$

which yields

$$\lambda = \pm\sqrt{k^2 - \rho}$$

for small enough k, the roots are always imaginary. However, for large enough k, one root will always exist that causes the solution to grow exponentially.

Definition 2 (Stability)**.** *A normal mode solution to a linear PDE with constant coefficients is called unstable for wavenumber k if there exists a root with* $\mathrm{Re}\,(\lambda(\mathrm{k})) > 0$*. If all roots have the property that* $\mathrm{Re}\,(\lambda(\mathrm{k})) < 0$*, then the solution is said to be asymptotically stable. Finally, if all roots have the property that* $\mathrm{Re}\,(\lambda(\mathrm{k})) \leq 0$ *then the solution is said to be stable.*