1 Normal Mode Solutions

Up until now we've only focused on unit amplitude solutions, if instead we were to consider

$$u(x,t) = \frac{1}{\lambda(k)^2}\,\exp\left[i\,k\,x + \lambda(k)\,t\right] \tag{1}$$

then for $\lambda = k$, this function is a solution of Laplace's equation. This solution has initial values

$$u(x,0) = \frac{1}{\lambda(k)^2}\exp[i\,k\,x], \quad u_t(x,0) = \frac{1}{\lambda(k)}\exp[i\,k\,x].$$

Here note that since $\lambda \to \infty$ as $k \to \infty$, then the ICs can be made to be arbitrarily small by picking appropriate k.
Now let's suppose we had two IVPs:

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} = 0, \quad u(x,0) = f(x), \quad u_t(x,0) = g(x)$$

and

$$\frac{\partial^2 v}{\partial t^2} + \frac{\partial^2 v}{\partial x^2} = 0, \quad v(x,0) = f(x) + \epsilon_1(x), \quad v_t(x,0) = g(x) + \epsilon_2(x)$$

with solutions u and v. Don't worry about the form of ϵs, instead just think of them as small deviations. If we recall M3 from Hadamard's notions of well posedness, then we'd expect u and v to be close to one another. Let's make this a bit more formal, take $w = v - u$. A math-ier way of saying "u and v stay close to one another" is to say that w stays bounded. Well, w satisfies the IVP

$$\frac{\partial^2 w}{\partial t^2} + \frac{\partial^2 w}{\partial x^2} = 0, \quad w(x,0) = \epsilon_1(x), \quad w_t(x,0) = \epsilon_2(x)$$

If we take w as in (1), then w grows without bound no matter how small the initial data is. Motivating...

Theorem 1. *Let*

$$\Omega = \sup_{k\in\mathbb{R}}\left(\mathrm{Re}(\lambda(k))\right)$$

then, if

- $\Omega = \infty$*, the PDE is not well posed*
- $\Omega \in (0,\infty)$*, the PDE is well posed, but is unstable*
- $\Omega = 0$*, the PDE is well posed and stable*
- $\Omega \in (-\infty, 0)$*, the PDE is well posed and asymptotically stable*

1.1 Dispersion

We saw what it meant for constant coefficient equations to permit wave-like solutions and what we mean by stability in that sense, now we will talk about the speed of these wave like solutions.
To that end, we focus on a particular version of the normal mode solution of the form

$$u(x,t) = A(k)\,\exp[i\,\theta_k(x,t)], \quad \text{where } \theta_k(x,t) = k\,x - \omega(k)\,t$$

where k is the wavenumber, θ_k is the phase of the wave, and ω is the frequency of the wave. (Note that this is just as general as the previous form of the normal mode solution, we're just writing λ slightly differently and naming it ω)

If we wanted to find the speed of the wave, then we could trace the position of the peak of the wave. If we're staying on exactly the peak, then the phase is constant. i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\theta = 0 = k\,x'(t) - \omega \implies x'(t) = \frac{\omega}{k}.$$

we call ω/k the phase-speed of the wave. To determine this phase speed, we first need to find ω (like how we found λ before). Hence to find ω we seek to construct a **dispersion relation** by substituting this form of u into the PDE and cancelling the exponentials.

Example 1. *Find a dispersion relation for the Klein-Gordon Equation*

$$u_{tt} - c^2\,u_{xx} + \alpha^2\,u = 0$$

Substituting $u(x,t) = \exp[i\,(k\,x - \omega\,t)]$ into the PDE yields the dispersion relation

$$\begin{aligned}(-i\,\omega)^2 - c^2\,(i\,k)^2 + \alpha^2 &= 0\\ -\omega^2 + c^2\,k^2 + \alpha^2 &= 0\end{aligned}$$

with solutions

$$\omega = \pm\sqrt{\alpha^2 + c^2\,k^2}.$$

(Note: the equation, $-\omega^2 + c^2\,k^2 + \alpha^2 = 0$ is called the dispersion relation for this PDE).
If $\alpha = 0$, then the PDE reduces to the wave equation. In that case, $\omega/k = \pm c$, hence the wave propagates to the left or right at a rate of c: something we already knew. If $\alpha \neq 0$, then the waves are propagating faster. In particular,

$$\frac{\omega}{k} = \pm\sqrt{c^2 + \frac{\alpha^2}{k^2}}.$$

Hence, short waves (large k) have waves with a speed near c, however longer waves (small k) can become arbitrarily fast.

2 Uniqueness of Solutions to the Wave Equation

We've found a solution to an IVP of the wave equation via the method of characteristics: namely, d'Alembert's solution. Hence we've addressed M1 in this context. We saw that normal mode solutions of the equation are stable, hence the equation is well-posed (see previous Theorem). Now we will answer the question of M2: is d'Alembert's solution unique? In so doing, we will build a general approach for answering questions of uniqueness via Energy functions.
Suppose u_1 and u_2 satisfy the wave equation

$$\frac{\partial^2 u_i}{\partial t^2} - c^2\frac{\partial^2 u_i}{\partial x^2} = 0, \quad u_i(x,0) = f(x), \quad \frac{\partial}{\partial t}u_i(x,0) = g(x), \quad -\infty < x < \infty, \quad t \geq 0, \quad i = 1,2$$

then let $v = u_1 - u_2$. Hence v satisfies the IVP

$$\frac{\partial^2 v}{\partial t^2} - c^2\frac{\partial^2 v}{\partial x^2} = 0, \quad v(x,0) = 0, \quad \frac{\partial}{\partial t}v(x,0) = 0, \quad -\infty < x < \infty, \quad t \geq 0.$$

Recall, d'Alembert's solution is

$$v(x,t) = \frac{u_0(x - c\,t) + u_0(x + c\,t)}{2} + \frac{1}{2c}\int_{x-c\,t}^{x+c\,t} v_0(s)\,\mathrm{d}s$$

Now d'Alembert's solution gives us a way to go from Cauchy data to solution for the wave equation, if we were to do this for the IVP for v we would find that d'Alembert's solution predicts $v(x,t) = 0$. If $v = 0$ is the *only* solution, then we've proven that d'Alembert's general solution is unique.

We form an energy function as follows:

$$\begin{aligned}
v_{tt} &= c^2\, v_{xx} \\
v_t\, v_{tt} &= c^2\, v_t\; v_{xx} \\
\frac{\partial}{\partial t}\left(\frac{1}{2}\, v_t^2\right) &= c^2\, v_t\, v_{xx} \\
\frac{\partial}{\partial t}\left(\frac{1}{2}\, v_t^2\right) &= \frac{\partial}{\partial x}\left(c^2 v_t\, v_x\right) - c^2\, v_{tx}\, v_x \\
\frac{\partial}{\partial t}\left(\frac{1}{2}\, v_t^2\right) &= \frac{\partial}{\partial x}\left(c^2 v_t\, v_x\right) - \frac{\partial}{\partial t}\left(\frac{c^2}{2}\, v_x^2\right) \\
\frac{\partial}{\partial t}\left(\frac{1}{2}\, v_t^2 + \frac{c^2}{2}\, v_x^2\right) &= \frac{\partial}{\partial x}\left(c^2 v_t\, v_x\right)
\end{aligned}$$

We then integrate with respect to x making the (reasonable) assumption that displacements vanish at $\pm\infty$ we see

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\partial}{\partial t}\left(\frac{1}{2}\, v_t^2 + \frac{c^2}{2}\, v_x^2\right)\, \mathrm{d}x &= \int_{-\infty}^{\infty} \frac{\partial}{\partial x}\left(c^2 v_t\, v_x\right)\, \mathrm{d}x \\
\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} \left(\frac{1}{2}\, v_t^2 + \frac{c^2}{2}\, v_x^2\right)\, \mathrm{d}x &= \left[c^2 v_t\, v_x\right]_{x=-\infty}^{\infty} \\
\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} \left(\frac{1}{2}\, v_t^2 + \frac{c^2}{2}\, v_x^2\right)\, \mathrm{d}x &= 0
\end{aligned}$$

For ease of notation, let

$$E(t) = \int_{-\infty}^{\infty} \left(\frac{1}{2} v_t^2 + \frac{1}{2} c^2\, v_x^2\right)\, \mathrm{d}x$$

then we have determined that

$$E'(t) = 0.$$