1 Energy Functions

Previously we defined the following energy function

$$E(t) = \int_{-\infty}^{\infty} \left(\frac{1}{2} v_t^2 + \frac{1}{2} c^2\, v_x^2 \right) \mathrm{d}x$$

and discovered that

$$\frac{\mathrm{d}E}{\mathrm{d}t}(t) = 0.$$

Before we continue, let's interpret what this function is saying. Focusing on the integrand, the term

$$\frac{1}{2} v_t^2$$

represents the kinetic energy of the system. Similarly,

$$\frac{1}{2} c^2\, v_x^2$$

is proportional to the potential energy of the system. (To motivate this, notice that the v_x is only non-zero when the string is not flat, hence v_x^2 is proportional to the potential energy). The entire integrand then is called the *energy density*, which is total energy per unit length. Hence $E(t)$ is then the total energy over the whole string (since we're integrating over the entire x domain). The statement $E'(t) = 0$ then means that energy is conserved in this system.

Now we want to use this energy function to show that $v = 0$, to do so we'll first show that $E(t) = 0$ for all t.

$$E(0) = \int_{-\infty}^{\infty} \left(\frac{1}{2} v_t(x,0)^2 + \frac{1}{2} c^2\, v_x(x,0)^2 \right) \mathrm{d}x$$

Now $v_t(x, 0)$ is given via our ICs as being identically zero hence the first term in the integrand is zero. Similarly, $v(x, 0)$ is identically 0 via our ICs as well. Hence $v_x(x, 0) = 0$ also (the string is initially flat). All told this tells us that $E(0) = 0$. Then, since $E'(t) = 0$, we have that $E(t) = 0$ for all time t.

Note, however, that the integrand of $E(t)$ is non-negative. Hence the only way that we can add up a bunch non-negative stuff and attain zero is if the integrand was uniquely zero itself. That is to say,

$$\frac{1}{2} v_t^2 + \frac{1}{2} c^2\, v_x^2 = 0, \quad \text{for all } t,\ x$$

and since both terms are individually non-negative, we have that

$$v_t = v_x = 0 \quad \text{for all } t,\ x.$$

Finally, since $v(x, 0) = 0$ and $v_t(x, t) = 0$ for all t and x we have that $v(x, t) = 0$ for all t and x. However,

$$v(x,t) = u_1(x,t) - u_2(x,t) = 0 \implies u_1(x,t) = u_2(x,t).$$

Therefore solutions to the wave equation with Cauchy initial data are unique.

2 IBVPs in Bounded Spatial Domains

For the next handful of weeks we will be restricting ourselves specifically to the problem of IVPs in Bounded Domains. Bounded domain problems are of a great deal of interest in the natural sciences and engineering for a variety of reasons, but not least of which is that the most powerful numerical methods are developed for bounded domains. We will now develop a general form of the types of PDEs we will

be considering. We'll develop theory for these general, heat-like governing equations and then apply those techniques in particular examples.
To start, we consider the conservation law where $\vec{\nabla}$ is only in terms of spatial derivatives

$$w(\vec{x})\,\frac{\partial u}{\partial t} + \vec{\nabla}\cdot\vec{\phi}(\vec{x},t) = f(\vec{x},t,u)$$

where we take a particular form of $\vec{\phi}$ informed via Fick's law as

$$\vec{\phi}(\vec{x},t) = -p(\vec{x})\,\vec{\nabla}\,u(\vec{x},t)$$

where p is like my diffusion parameter, just it's thought of as being able to vary in space.
For the form of f, the source term, use Newton's law of cooling which posits that the source is just a linear function of the solution. Hence we choose,

$$f(\vec{x},t,u) = -q(\vec{x})\,u(\vec{x},t) + w(\vec{x})\,F(\vec{x},t)$$

after which the PDE becomes

$$w\,\frac{\partial u}{\partial t} - \vec{\nabla}\cdot\left(p\,\vec{\nabla}\,u\right) + q\,u = w\,F.$$

We will focus on solving PDEs of this form, for which there are many, hence we define the following linear operator for shorthand

$$\mathscr{L}[u] = -\vec{\nabla}\cdot\left(p\,\vec{\nabla}u\right) + q\,u$$

Note: we assume that $w(\vec{x}) > 0$, $p(\vec{x}) > 0$, and $q(\vec{x}) \geq 0$. We dfine V as the bounded $\vec{x}$ domain. Interestingly, $\mathscr{L}$ is independent of time t.
Our general PDE then becomes

$$w\,\frac{\partial u}{\partial t} + \mathscr{L}[u] = w\,F$$

Now $\mathscr{L}$ is only in terms of space, so in 1D the class of this PDE is parabolic (there are no temporal derivatives hiding in $\mathscr{L}$, so there is no contribution to A and B from $\mathscr{L}$. There is also no contribution to A or B from the rest of the PDE, hence $A = B = 0$ and so the PDE is parabolic). We also consider related 1D hyperbolic and elliptic PDEs that involve $\mathscr{L}$.
For the hyperbolic case we consider

$$w\frac{\partial^2 u}{\partial t^2} + \mathscr{L}[u] = w\,F$$

and for the elliptic case we consider

$$w\frac{\partial^2 u}{\partial y^2} - \mathscr{L}[u] = 0$$

a homogeneous DE (in contrast with the inhomogeneous hyperbolic and parabolic cases).
The general boundary conditions we will consider are of the form

$$\alpha(\vec{x})\,u + \beta(\vec{x})\,\frac{\partial u}{\partial n}\Big|_{\partial V} = B(\vec{x},t)$$

where $x \in \partial V$ (the boundary of the spatial domain) and $\partial/\partial n$ refers to an outward facing normal derivative. This form of the BCs includes our Dirichlet conditions (take $\beta = 0$) and Neumann conditions (take $\alpha = 0$) of both homogeneous and inhomogeneous type (depending on B).
In 1D with $V = [0,l]$ this reduces to

$$\alpha_1\,u(0,t) - \beta_1\,u_x(0,t) = B_1(t)$$
$$\alpha_2\,u(l,t) + \beta_2\,u_x(l,t) = B_2(t)$$

we will use these same BCs for the hyperbolic, parabolic, and elliptic PDEs we consider in this section of the course. We will need more ICs for each particular class, the hyperbolic class has two temporal derivatives and so we'll need two ICs.

For the hyperbolic PDE we consider

$$u(\vec{x},0) = f(\vec{x}), \quad u_t(\vec{x},0) = g(\vec{x})$$

The parabolic PDE has only one temporal derivative, and so we need one IC

$$u(\vec{x},0) = f(\vec{x}).$$

The elliptic PDE has no temporal derivatives (and no time dependence in general) and so we provide no ICs.

Note that this general form of the PDE extends our prototypical examples. That is, if we take $w = 1$, $p = 1$, $q = 0$, $F = 0$ the parabolic PDE reduces to the heat equation, the elliptic PDE reduces to Laplace's equation, and the hyperbolic PDE reduces to the wave equation.