1 Separation of Variables

To motivate this section, let's start with an example.

Example 1. *Solve the following heat equation*

$$\frac{\partial u}{\partial t} = D\,\frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = u(l,t) = 0, \quad u(x,0) = f(x)$$

by assuming that $u(x,t) = X(x)\,T(t)$.

First note that the trivial solution $u = 0$ solves the BVP but doesn't (in general) satisfy the IC. Let's suppose $u \neq 0$ to try and find more general, non-trivial solutions.
As suggested, we guess that $u(x,t) = X(x)\,T(t)$ (we are not justified in this guess just yet, but if we manage to solve the equation this way and prove uniqueness, then we'll have justified that we didn't lose any generality).
Trying to make our guess fit:

$$\begin{aligned} u_t &= D\,u_{xx} \\ X(x)\,T'(t) &= D\,X''(x)\,T(t) \\ \frac{1}{D}\frac{T'(t)}{T(t)} &= \frac{X''(x)}{X(x)} \end{aligned}$$

this equation must be true for all x and t. But, now note that the LHS of this is independent of space and the RHS of this is independent of time. Hence the LHS is constant in space and the RHS is constant in time. But the two are equal to one another, so they must be equal to the same constant. Hence,

$$\frac{1}{D}\frac{T'(t)}{T(t)} = \lambda = \frac{X''(x)}{X(x)}$$

(This constant, λ, is sometimes called the separation constant. We'll see how we can think of it as an eigenvalue, justifying the choice of the letter λ).
Or, in other words,

$$T'(t) - D\,\lambda\,T(t) = 0 \tag{1}$$

$$X''(x) - \lambda\,X(x) = 0 \tag{2}$$

and both of these are ODEs! Now let's consider our BCs (we'll worry about IC later).

$$u(0,t) = 0 = X(0)T(t)$$

so either $X(0) = 0$ or $T(t) = 0$. Note if $T(t) = 0$, then u is the trivial solution. Hence we take $X(0) = 0$.
Similarly, the $u(l,t) = 0 = X(l)T(t)$ boundary condition gives us $X(l) = 0$.
Equation (1) is easily solved (separable ODE) to get

$$T(t) = A\,e^{D\,\lambda\,t}$$

for some value of A.
Let's solve equation (2). The characteristic polynomial of this equation is

$$r^2 - \lambda = 0$$

and hence the solution depends on the sign of λ. We will proceed case-wise,

1. $\lambda = 0$, then $X''(x) = 0 \implies X(x) = c_1\,x + c_2$. Then $X(0) = c_2 = 0$ and $X(l) = c_1\,l = 0 \implies c_1 = 0$. Hence $X(x) = 0$ and so $u(x,t) = 0$. But we wanted to avoid the trivial case, so we must require that $\lambda \neq 0$.

2. $\lambda > 0$, equation (2) has solution $X(x) = c_1\,e^{\sqrt{\lambda}\,x} + c_2\,e^{-\sqrt{\lambda}\,x}$. Hence $X(0) = c_1 + c_2 = 0 \implies c_1 = -c_2$ and $X(l) = c_1\,e^{\sqrt{\lambda}\,l} - c_1\,e^{-\sqrt{\lambda}\,l} = 0 \implies c_1\,(e^{2\sqrt{\lambda}\,l} - 1) = 0 \implies c_1 = 0$. And so the trivial solution happens here as well. (We could have done the last two cases in one shot by taking $\lambda \geq 0$, but I did it slower for illustrative purposes).

3. $\lambda < 0$, equation (2) has solution $X(x) = c_1\,\sin(\sqrt{-\lambda}\,x) + c_2\,\cos(\sqrt{-\lambda}\,x)$. Applying the BCs yields $X(0) = c_2 = 0$ and $X(l) = c_1\,\sin(\sqrt{-\lambda}\,l) = 0$. Which is true if either $c_1 = 0$ or $\sqrt{-\lambda}\,l = k\,\pi$ where $k \in \mathbb{Z}$. If $c_1 = 0$, then we have the trivial solution again so we must have that $\lambda = -\left(\frac{k\pi}{l}\right)^2$. In which case $X(x) = c_1\,\sin(|k|\,\pi\,l^{-1}\,x)$. So if $k = 0$, then the trivial solution arises again. Hence, without loss of generality, take $X(x) = c_1\,\sin(k\,\pi\,l^{-1}\,x)$ for $k = 1, 2, \ldots$

Hence our only non-trivial solution in this approach is given by

$$u_k(x,t) = X_k(x)\,T_k(t) = c_k\,e^{-\frac{k^2\,\pi^2}{l^2}\,D\,t}\,\sin\left(x\,\frac{k\,\pi}{l}\right)$$

where I've written the constant c_1 as c_k to emphasize that the constant can change for any k value we consider. Similarly, the k subscripts in u, X, and T are to emphasize that this is a solution for all values of positive integer k.

Now let's apply the IC:

$$u_k(x,0) = f(x) = c_k\,\sin\left(x\frac{k\,\pi}{l}\right).$$

It feels like we've made a mistake. The RHS of the above only depends on x inside the sine, so if $f(x)$ is a general function we cannot find a value of c_k to make this relation true. However, the heat equation is linear and u_k is a solution for all $k = 1, 2, \ldots$. We take

$$u(x,t) = \sum_{k=1}^{\infty} c_k\,\sin\left(x\,\frac{k\,\pi}{l}\right)\,e^{-\frac{k^2\,\pi^2}{l^2}\,D\,t} \tag{3}$$

and *now* apply the IC to see

$$u(x,0) = \sum_{k=1}^{\infty} c_k\,\sin\left(x\,\frac{k\,\pi}{l}\right)$$

which should look an awful lot like the Fourier sine series. Hence we choose the form of c_k to be the Fourier sine coefficients of $f(x)$.

2 Separating our 1D (Homogeneous) General Equations

Starting with the homogeneous (no F) forms of our general equations, beginning with our hyperbolic equation

$$\begin{aligned} w(x)\,\frac{\partial^2 u}{\partial t^2} &= -\mathscr{L}[u] \\ w(x)\,X(x)\,T''(t) &= -T(t)\mathscr{L}[X(x)] \\ \frac{T''(t)}{T(t)} &= -\frac{1}{w(x)}\,\frac{\mathscr{L}[X(x)]}{X(x)} \end{aligned}$$

where my LHS is only in terms of time and my RHS in terms of space, again. Similarly, for our parabolic equation.

$$\begin{aligned} w(x)\,\frac{\partial u}{\partial t} &= -\mathscr{L}[u] \\ \frac{T'(t)}{T(t)} &= -\frac{1}{w(x)}\,\frac{\mathscr{L}[X(x)]}{X(x)} \end{aligned}$$

For separating the elliptic equation we adopt the notation $u(x,y) = X(x)\,Y(y)$ instead to see

$$\begin{aligned} -w(x)\,\frac{\partial^2 u}{\partial y^2} &= -\mathscr{L}[u] \\ -\frac{Y''(y)}{Y(y)} &= -\frac{1}{w(x)}\frac{\mathscr{L}[X(x)]}{X(x)} \end{aligned}$$

If we take the separation constant as $-\lambda$ (here I'm *not yet* assuming the sign, λ is arbitrarily signed). The left hand sides of the three equations then become

$$\begin{aligned} T''(t) + \lambda\,T(t) &= 0, \quad \text{hyperbolic} \\ T'(t) + \lambda\,T(t) &= 0, \quad \text{parabolic} \\ Y''(y) - \lambda\,Y(y) &= 0, \quad \text{elliptic.} \end{aligned}$$

For all three cases the RHS is the same. Then, we get

$$\begin{aligned} -\frac{1}{w(x)}\frac{\mathscr{L}[X(x)]}{X(x)} &= -\lambda \\ \frac{1}{w(x)}\mathscr{L}[X(x)] &= \lambda\,X(x) \qquad (4) \\ \mathscr{L}[X(x)] &= \lambda\,w(x)\,X(x) \end{aligned}$$

The final equation is probably the most algebraically convenient, but the middle equation (4) might be the most enlightening. In particular, (4) motivates *why* we call λ an eigenvalue. λ is then an eigenvalue and $X(x)$ is an eigenfunction for the operator $\frac{1}{w}\mathscr{L}$.