We ended last time by discussing how linear operators give rise to eigenvalues in a way similar to the eigenvalues you see in a linear algebra class. We will now begin with some definitions similar to what you've seen in a linear algebra class:

Definition 1. *The* ***inner product*** *of two functions f and g with weight $w(x) > 0$ is defined as*

$$(f, g) = \int_V w(\vec{x})\, f(\vec{x})\, g(\vec{x})\, \mathrm{d}V.$$

In 1D for complex f and g we call

$$(f, g) = \int_V w(x)\, f(x)\, \bar{g}(x)\, \mathrm{d}x$$

the ***Hermitian*** *inner product where $\bar{g}$ refers to the complex conjugate of g.*

Note that the inner product is symmetric since $(f, g) = (g, f)$ and is bi-linear (i.e. linear in both arguments: $(f + g, h) = (f, h) + (g, h)$ and $(f, g + h) = (f, g) + (f, h)$). (The Hermitian inner product is not symmetric in general).

Definition 2. *If $(f, g) = 0$ then f and g are called* ***orthogonal****.*

Definition 3. *The* ***norm*** *of a function f is then defined as*

$$\|f\| = \sqrt{(f, f)}$$

Note that since $w(x) > 0$ we have that $\|f\| \geq 0$. Moreover $\|f\| = 0$ if, and only if, $f = 0$. If the norm of a function is finite on V, then the function is called **square integrable** on V.

Definition 4. *If f is a square integrable function, then*

$$\frac{f}{\|f\|}$$

is called ***normalized****.*

Definition 5. *A differential operator L is called* ***self-adjoint*** *if*

$$(v, L[u]) = (L[v], u)$$

for any two smooth functions v and u. (Verifying this requires boundary conditions!)

Definition 6. *A set of functions $\{f_k(x)\}_{k \in \mathbb{Z}_{>0}}$ is called an* ***orthogonal*** *set if $(f_i, f_j) = 0$ for all $i \neq j$. Additionally, if each f_i is normalized, then the set is called an* ***orthonormal*** *set.*

Example 1. *Verify that $\frac{1}{w(\vec{x})}\mathscr{L}$ is self-adjoint for homogeneous BCs*

A-propos of nothing I'm going to start with the BCs:

$$\alpha(\vec{x})\, u + \beta(\vec{x}) \left.\frac{\partial u}{\partial n}\right|_{\partial V} = 0$$

hence

$$\left.\frac{\partial u}{\partial n}\right|_{\partial V} = -\frac{\alpha(\vec{x})}{\beta(\vec{x})}\, u.$$

Now, recall that $f\, \vec{\nabla} \cdot \vec{g} = \vec{\nabla} \cdot (f\, \vec{g}) - \vec{\nabla} f \cdot \vec{g}$.

Let u and v be two smooth functions. To start, we consider

$$\begin{aligned} v\,\mathscr{L}[u] - u\,\mathscr{L}[v] &= v\,\left(-\vec{\nabla}\cdot(p\,\vec{\nabla}\,u) + q\,u\right) - u\left(-\vec{\nabla}\cdot(p\vec{\nabla}v) + q\,v\right) \\ &= -v\,\vec{\nabla}\cdot(p\,\vec{\nabla}\,u) + u\,\vec{\nabla}\cdot(p\vec{\nabla}v) \\ &= -\vec{\nabla}\cdot(p\,v\,\vec{\nabla}\,u) + \vec{\nabla}v\cdot(p\,\vec{\nabla}u) + \vec{\nabla}\cdot(p\,u\vec{\nabla}v) - \vec{\nabla}u\,\cdot(p\vec{\nabla}v) \\ &= -\vec{\nabla}\cdot\left(p\,\left(v\,\vec{\nabla}u - u\vec{\nabla}v\right)\right) \end{aligned}$$

This next step illustrates *why* we care about self-adjoint operators

$$\begin{aligned} \int_V v\,\mathscr{L}[u] - u\,\mathscr{L}[v]\,\mathrm{d}V &= \int_V -\vec{\nabla}\cdot\left(p\,\left(v\,\vec{\nabla}u - u\vec{\nabla}v\right)\right)\,\mathrm{d}V \\ &= -\int_{\partial V}\left(p\,\left(v\,\vec{\nabla}u - u\vec{\nabla}v\right)\right)\cdot\vec{n}\,\mathrm{d}A \\ &= -\int_{\partial V} p\,\left(v\,\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right)\,\mathrm{d}A \end{aligned}$$

and now I sub in my BCs to see (which I can do since the integral is over the boundary)

$$\int_V v\,\mathscr{L}[u] - u\,\mathscr{L}[v]\,\mathrm{d}V = -\int_{\partial V} p\,\left(-v\,\frac{\alpha}{\beta}\,u + u\,\frac{\alpha}{\beta}\,v\right)\,\mathrm{d}A = 0$$

and finally, to wrap up, I note that

$$\int_V v\,\mathscr{L}[u] - u\,\mathscr{L}[v]\,\mathrm{d}V = \int_V w\,\left(v\,\frac{1}{w}\,\mathscr{L}[u] - u\,\frac{1}{w}\,\mathscr{L}[v]\right)\,\mathrm{d}V = \left(v,\,\frac{1}{w}\,\mathscr{L}[u]\right) - \left(u,\,\frac{1}{w}\,\mathscr{L}[v]\right) = 0$$

Hence, $\frac{1}{w(\vec{x})}\mathscr{L}$ is a self-adjoint operator.

Definition 7. *If u is a smooth function with $u \neq 0$, then L is called* ***positive semi-definite*** *if*

$$(u, L[u]) \geq 0$$

and ***positive definite*** *if*

$$(u, L[u]) > 0.$$

Example 2. *Show that $\frac{1}{w}\mathscr{L}$ is positive definite for homogeneous boundary conditions with $\alpha \geq 0$ and $\beta \geq 0$.*

$$\begin{aligned} \left(u,\,\frac{1}{w}\mathscr{L}[u]\right) &= \int_V \left(-u\vec{\nabla}\cdot(p\,\vec{\nabla}u) + q\,u^2\right)\,\mathrm{d}V \\ &= \int_V \left(-\vec{\nabla}\cdot(p\,u\,\vec{\nabla}u) + p\,(\vec{\nabla}u)^2 + q\,u^2\right)\,\mathrm{d}V \\ &= \int_V \left(p\,(\vec{\nabla}u)^2 + q\,u^2\right)\,\mathrm{d}V - \int_{\partial V} p\,u\,\frac{\partial u}{\partial n}\,\mathrm{d}A \\ &= \int_V \left(p\,(\vec{\nabla}u)^2 + q\,u^2\right)\,\mathrm{d}V + \int_{\partial V} p\,u^2\,\frac{\alpha}{\beta}\,\mathrm{d}A \geq 0 \end{aligned}$$

since $p > 0$, $q \geq 0$, $\alpha \geq 0$, $\beta \geq 0$ by assumption. Moreover, we only achieve 0 when u is itself the trivial solution. Hence $1/w\mathscr{L}$ is a positive definite operator.
Note: in the above calculation I used the *slight* abuse of notation

$$(\vec{\nabla}u)^2 = \vec{\nabla}u\cdot\vec{\nabla}u$$