Many of the properties we saw last time depend upon our boundary conditions. Moreover, we built all these properties to help us generalize some linear algebra stuff so that we could find eigenvalues.
For a general linear operator L, the problem of finding u and λ such that

$$L[u] = \lambda\, u$$

given BCs is called an *eigenvalue-problem.* A special type of eigenvalue problem crops up a lot in the study of differential equations. In this course, we will focus on the one-dimensional eigenvalue problem

$$\frac{1}{w(x)}\mathscr{L}[u] = -\frac{1}{w(x)}\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\,\frac{\mathrm{d}u}{\mathrm{d}x}\right) + \frac{q(x)\,u(x)}{w(x)} = \lambda u(x)$$

on $0 < x < l$ with the BCs

$$\alpha_1\, u(0) - \beta_1\, u'(0) = 0, \quad \alpha_2\, u(l) + \beta_2\, u'(l) = 0.$$

This ODE and two BCs are collectively called a *Sturm-Liouville problem* **if**

1. $p(x) > 0$, $w(x) > 0$, and $q(x) \geq 0$
2. p, q, w, and p' are continuous on $[0, l]$

These conditions are enough to guarantee uniqueness of solutions to the Sturm-Liouville problem. Finally, if $\alpha_i \geq 0$, $\beta_i \geq 0$, and $\alpha_i + \beta_i > 0$, then we call it a *Regular* Sturm-Liouville problem. Otherwise, it is called a *Singular* Sturm-Liouville problem.
Let's see what we can deduce about the solutions to the regular Sturm-Liouville problem: but first, note that since w, p, and q are real we have that

$$\overline{\frac{1}{w}\mathscr{L}[u]} = \frac{1}{w}\mathscr{L}[\overline{u}].$$

Now,

Property 1 Eigenfunctions of differing eigenvalues are orthogonal

Suppose λ_i and λ_j are two distinct eigenvalues with u_i and u_j as eigenfunctions. Then,

$$\begin{aligned}\left(u_i, \frac{1}{w}\mathscr{L}[u_j]\right) - \left(\frac{1}{w}\mathscr{L}[u_i], u_j\right) &= \int_0^l w\,\left(u_i\,\overline{\frac{1}{w}\mathscr{L}[u_j]} - \frac{1}{w}\mathscr{L}[u_i]\,\overline{u_j}\right)\,\mathrm{d}x \\ &= \int_0^l w\,\left(u_i\,\lambda_j\,\overline{u_j} - \lambda_i\,u_i\,\overline{u_j}\right)\,\mathrm{d}x \\ &= (\lambda_j - \lambda_i)\int_0^l w\,u_i\,\overline{u_j}\,\mathrm{d}x \\ &= (\lambda_j - \lambda_i)\,(u_i, u_j) = 0\end{aligned}$$

Hence since $\lambda_j \neq \lambda_i$, we have that $(u_i, u_j) = 0$.

Property 2 Eigenvalues are real and non-negative and eigenfunctions can be taken as real valued WLOG.

Suppose λ_i and u_i are complex valued. Then,

$$\frac{1}{w}\mathscr{L}[u_i] = \lambda_i\,u_i \implies \overline{\frac{1}{w}\mathscr{L}[u_i]} = \overline{\lambda_i}\,\overline{u_i} \implies \frac{1}{w}\mathscr{L}[\overline{u_i}] = \overline{\lambda_i}\overline{u_i}$$

That is, $\overline{\lambda_i}$ is an eigenvalue to $\overline{u_i}$.

Hence,

$$
\begin{aligned}
\left(u_i, \frac{1}{w}\,\mathscr{L}[u_i]\right) - \left(\frac{1}{w}\,\mathscr{L}[u_i], u_i\right) &= 0 \\
(u_i, \lambda_i\, u_i) - (\lambda_i\, u_i, u_i) &= 0 \\
\overline{\lambda_i}\,(u_i, u_i) - \lambda_i\,(u_i, u_i) &= 0 \\
(\overline{\lambda_i} - \lambda_i)\,\|u_i\|^2 &= 0
\end{aligned}
$$

Since the eigenvalue is non-trivial, we have that $\overline{\lambda_i} - \lambda_i = 0$. Thus λ_i is purely real.

For non-negativity, notice that

$$
\begin{aligned}
\left(u, \frac{1}{w}\mathscr{L}[u]\right) &= -\int_0^l u\,\frac{\mathrm{d}}{\mathrm{d}x}(p\,u')\,\mathrm{d}x + \int_0^l (q\,u^2)\,\mathrm{d}x \\
&= -\int_0^l \frac{\mathrm{d}}{\mathrm{d}x}\,(u\,p\,u') - p\,u'^2\mathrm{d}x + \int_0^l (q\,u^2)\,\mathrm{d}x \\
&= -[p\,u\,u']_0^l + \int_0^l p\,u'^2\,\mathrm{d}x + \int_0^l q\,u^2\,\mathrm{d}x
\end{aligned}
$$

the final two terms of which are evidently non-negative. For the first term notice that

$$
-[p\,u\,u']_0^l = -p(l)\,u(l)\,u'(l) + p(0)\,u(0)\,u'(0).
$$

Recall the boundary condition

$$
\alpha_2\,u(l) + \beta_2\,u'(l) = 0
$$

then, if $\beta_2 = 0$ we must have $u(l) = 0$, otherwise

$$
u'(l) = -\frac{\alpha_2}{\beta_2}\,u(l)
$$

hence

$$
-p(l)\,u(l)\,u'(l) = p(l)\,(u(l))^2\,\frac{\alpha_2}{\beta_2} \geq 0
$$

Similarly, if $\beta_1 = 0$, then $u(0) = 0$, otherwise

$$
p(0)\,u(0)\,u'(0) = p(0)\,(u(0))^2\,\frac{\alpha_1}{\beta_1} \geq 0.
$$

In any case, we have that

$$
\left(u, \frac{1}{w}\mathscr{L}[u]\right) \geq 0
$$

However,

$$
0 \leq \left(u, \frac{1}{w}\mathscr{L}[u]\right) = (u, \lambda\,u) = \lambda\,(u, u) = \lambda\,\|u\|^2
$$

Hence, since $\|u\| > 0$ we have $\lambda \geq 0$.