1 Properties of the Regular Sturm-Liouville Problem

Property 2 Cont'd *Last time we saw that eigenvalues are real and non-negative, now we'll see that WLOG eigenfunctions are real.*

To start, suppose $u = u_R + i\, u_I$ is an eigenfunction to a real eigenvalue λ, then

$$\begin{aligned} L[u] &= \lambda\, u \\ L[u_R + i\, u_I] &= \lambda\,(u_R + i\, u_I) \\ L[u_R] + i\, L[u_I] &= \lambda\, u_R + i\, \lambda\, u_I \end{aligned}$$

Hence if we equate real parts of the equation and imaginary parts of the equation, we see that u_R and u_I are both eigenfunctions to λ. As a result we don't need to consider complex eigenfunctions: we need only find all the real valued eigenfunctions for an eigenvalue.

Property 3 Each eigenvalue has multiplicity one. That is, for each eigenvalue there is only one eigenfunction (up to linear dependence). Suppose this weren't the case, then there would be two solutions u_1 and u_2 to the same Sturm-Liouville ODE with the same BCs. Hence,

$$\alpha_2\, u_1(l) + \beta_2\, u_1'(l) = 0, \quad \alpha_2\, u_2(l) + \beta_2\, u_2'(l) = 0$$

In matrix notation,

$$\begin{bmatrix} u_1(l) & u_1'(l) \\ u_2(l) & u_2'(l) \end{bmatrix} \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix} = \vec{0}$$

Now $\alpha_2 + \beta_2 > 0$, hence we must have that the matrix has zero determinant. But the determinant of this matrix is the Wronskian of u_1 and u_2. Since u_1 and u_2 are solutions of a differential equation if their Wronskian vanishes anywhere, then the functions are linearly dependent.

Property 4 There is a countable infinity of eigenvalues with a point at infinity. That is,

$$0 \leq \lambda_1 < \lambda_2 < \cdots$$

and $\lambda_k \to \infty$ as $k \to \infty$.

This set of eigenvalues is called the **spectrum** of the differential operator.

Property 5 The eigenfunctions form an orthonormal set.

We already saw orthogonality in Property 1, for the normality of eigenfunctions consider that for any linear operator L,

$$L[\alpha\, u] = \alpha\, L[u].$$

If we take $\alpha = 1/\|u\|$ where u is part of the eigenvalue-eigenfunction pair (λ, u), then

$$L\left[\frac{u}{\|u\|}\right] = \frac{1}{\|u\|} L\,[u] = \frac{1}{\|u\|}\,\lambda\, u = \lambda\, \frac{u}{\|u\|}$$

and so we can normalize (or scale) any eigenfunction and it is still an eigenfunction with respect to the same eigenvalue.

Intuitively this orthonormal, infinite sized set of eigenfunctions acts like a basis for solutions of our PDE.

Definition 1 (Generalized Fourier Series). *For a set of square integrable functions $\{f_k(x)\}$ we define the Fourier coefficients of a square integrable function f to be*

$$a_k = (f, f_k)$$

and the (Generalized) Fourier Series as

$$\sum_{k=1}^{\infty} a_k\, f_k(x) \tag{1}$$

Note that if the f_ks are normalized (like eigenfunctions are), then $(f, f_k)f_k$ is the projection of f onto f_k. In the case that f_ks are eigenfunctions, we call equation (1) an *eigenfunction expansion.* For any series of functions it is important for us to ensure that the series converges. For equation (1), we want it to converge to our original function. To that end, we consider the partial sum

$$s_N(x) = \sum_{k=1}^{N} a_k\, f_k(x)$$

where here we assume that the f_ks form an orthonormal set. Then we calculate

$$\begin{aligned}
\|f - s_N\|^2 &= \left(f - \sum_{k=1}^{N} a_k\, f_k(x), f - \sum_{k=1}^{N} a_k\, f_k(x) \right) \\
&= \left(f, f - \sum_{k=1}^{N} a_k\, f_k(x) \right) - \left(\sum_{k=1}^{N} a_k\, f_k(x), f - \sum_{k=1}^{N} a_k\, f_k(x) \right) \\
&= (f, f) - \left(f, \sum_{k=1}^{N} a_k\, f_k(x) \right) - \left(\sum_{k=1}^{N} a_k\, f_k(x), f \right) + \left(\sum_{k=1}^{N} a_k\, f_k(x), \sum_{k=1}^{N} a_k\, f_k(x) \right) \\
&= \|f\|^2 - 2 \sum_{k=1}^{N} a_k\ (f, f_k) + \sum_{k=1}^{N} \sum_{j=1}^{N}\ a_k\, a_j\ (f_k, f_j) \\
&= \|f\|^2 - 2 \sum_{k=1}^{N} a_k^2 + \sum_{k=1}^{N} \sum_{j=1}^{N}\ a_k\, a_j\ (f_k, f_j) \\
&= \|f\|^2 - \sum_{k=1}^{N}\ a_k^2 \geq 0
\end{aligned}$$

Hence,

$$\sum_{k=1}^{N} (f, f_k)^2 \leq \|f\|^2$$

which is true for any value of N, hence

$$\sum_{k=1}^{\infty} (f, f_k)^2 \leq \|f\|^2$$

this is called Bessel's Inequality. (Note: subtle thing here, we know that the series converges to a number since we showed that it was bounded above. Since each term in the partial sum is evidently non-negative, the series defines a bounded, monotone increasing sequence of partial sums which are convergent via MCT. This just tells us that the series in Bessel's Inequality converges, not what it converges to).

When equality is reached, we call it Parseval's Identity:

$$\sum_{k=1}^{\infty} (f, f_k)^2 = \|f\|^2.$$

When Parseval's Identity holds we have that

$$\lim_{N\to\infty} \|f - s_N\| = 0$$

in which case we say s_N **converges** to f in the **mean**. This is called **mean square convergence**.

Definition 2. *A set of square integrable functions is said to be* ***complete*** *if for any square integrable function f, its generalized Fourier Series converges in the mean.*

Showing completeness is beyond the scope of this course. However, we don't need to worry about completeness in the context of Regular Sturm-Liouville problems because of our final property:

Property 6 The set of eigenfunctions forms a complete, orthonormal set of square integrable functions over the integral $0 < x < l$.

Now we'll see how we can use this theory to help us find eigenvalues and then to find separable solutions.