1 Sturm-Liouville and Separation of Variables

Recall from Wed June 1st (Lec 13), our original hyperbolic and parabolic equations were

$$w(x)\,\frac{\partial^2 u}{\partial t^2} + \mathscr{L}[u] = 0, \quad \text{hyperbolic}$$
$$w(x)\,\frac{\partial u}{\partial t} + \mathscr{L}[u] = 0, \quad \text{parabolic}$$

with homogeneous BCs:

$$\alpha_1\, u(0,t) - \beta_1\, u_x(0,t) = 0, \qquad \alpha_2\, u(l,t) + \beta_2\, u_x(l,t) = 0$$

and ICs $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$ when appropriate. We then separated the PDEs into the ODEs

$$\frac{1}{w(x)}\mathscr{L}[X](x) = \lambda_k\, X(x)$$

and

$$T''(t) + \lambda_k\, T(t) = 0, \quad \text{hyperbolic}$$
$$T'(t) + \lambda_k\, T(t) = 0, \quad \text{parabolic}$$

Now the λ_k is the same between the two, so we know that the λ_ks in this second set of ODEs are non-negative and countably-infinite.
For the hyperbolic and parabolic case our final solution will be of the form,

$$u(x,t) = \sum_{k=0}^{\infty} X_k(x)\, T_k(t)$$

where the X_ks are normalized eigenfunctions via the Sturm-Liouville problem, and the T_ks are from solving the second set of ODEs on the eigenvalues. As long as we can differentiate termwise, this satisfies the original PDE.
For the ICs, if we enforce that $T_k(0) = a_k = (f, X_k)$ as our IC for the separated ODE, we can take $u(x,0) = \sum_{k=0}^{\infty} a_k\, X_k(x) = f(x)$ to satisfy the IC for the PDE since the X_ks are complete. If we additionally require in the hyperbolic case that $T_k'(0) = b_k = (g, X_k)$, then we have that $u_t(x,0) = g(x)$. Hence the ICs are satisfied by solving an IVP in T.
For the BCs note that $u_x = \sum_{k=0}^{\infty} X_k'(x)\, T_k(t)$ hence

$$\alpha_1\, u(0,t) - \beta_1\, u_x(0,t) = \sum_{k=0^{\infty}} (\alpha_1\, X_k(0) - \beta_1\, X_k'(0))\, T_k(t) = 0$$

due to the BC in the Sturm-Liouville problem.
For the elliptic case we had

$$w(x)\,\frac{\partial^2 u}{\partial y^2} - \mathscr{L}[u] = 0$$

Which after taking $u(x,y) = X(x)\, Y(y)$ gave the SL problem and

$$Y''(y) - \lambda_k\, Y(y) = 0$$

as ODEs. If we consider the rectangular domain $V = [0 \ldots l_x] \times [0 \ldots l_y]$, with BCs

$$\alpha_1 u(0,y) - \beta_1\, u_x(0,y) = 0,\ \alpha_2 u(l_x,y) + \beta_2\, u_x(l_x,y) = 0, \quad u(x,0) = f(x),\ u(x,l_y) = g(x)$$

then the solution for X_k via the SL problem takes care of the x border's BCs. For the y borders, we have

$$u(x,0) = \sum_{k=0}^{\infty} X_k(x)\, Y_k(0)$$

so if $Y_k(0) = (f, X_k)$ then the $y = 0$ condition is satisfied. For $y = l_y$, we require that $Y_k(l_y) = (g, X_k)$. So to satisfy the y BC we must solve the BVP in Y. We should ensure this is always possible: if $\lambda_k \neq 0$, then

$$Y'' - \lambda_k\, Y \implies Y(y) = c_1 \exp(\sqrt{\lambda_k}\, y) + c_2 \exp(-\sqrt{\lambda_k}\, y)$$

Hence

$$\begin{bmatrix} Y(0) \\ Y(l) \end{bmatrix} = \begin{bmatrix} (f, X_k) \\ (g, X_k) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \exp(\sqrt{\lambda_k}\, l_y) & \exp(-\sqrt{\lambda_k}\, l_y) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

where the matrix is invertible for non-zero eigenvalues. If $\lambda = 0$, then the solution is

$$Y'' = 0 \implies Y = a\,x + b$$

hence by taking Y to be the straight line between (f, X_k) at $y = 0$ and (g, X_k) at $y = l_y$, the BVP is solved.

2 Examples

In general, we want to find λ-u pairs such that

$$\frac{1}{w}\mathscr{L}[u] = \lambda\, u, \quad \alpha_1\, u(0) - \beta_1\, u'(0) = 0, \quad \alpha_2\, u(l) + \beta_2\, u'(l) = 0.$$

The general process we will follow is to find v_1 and v_2 such that

$$\frac{1}{w}\mathscr{L}[v_i] = \lambda\, v_i, \quad v_1(0) = 1,\ v_1'(0) = 0, \quad v_2(0) = 0,\ v_2'(0) = 1$$

If we take $u(x) = \beta_1\, v_1(x) + \alpha_1\, v_2(x)$, then $u(0) = \beta_1$ and $u'(0) = \alpha_1$. Hence $\alpha_1\, u(0) - \beta_1\, u'(0) = \alpha_1\, \beta_1 - \beta_1\, \alpha_1 = 0$. For the second BC we need to use the values of λ to ensure it is satisfied. Our theory above tells us we will always be able to do this (and in so doing will find a countable infinite number of λs that make this possible). This is more clearly illustrated with an example.