Example 1. *Consider the linear operator*

$$L[u] = -\mu\, u''$$

under homogeneous Dirichlet BCs where $\mu \in \mathbb{R}_{>0}$.

a) *Show that L is self-adjoint and positive semi-definite*

The BCs correspond to $u(0) = u(l) = 0$. Hence $\alpha_1 = \alpha_2 = 1$ and $\beta_1 = \beta_2 = 0$. Now note that $\frac{1}{w}\mathscr{L} = L$ for $p(x) = \mu$, $w(x) = 1$, and $q(x) = 0$. We know that $\frac{1}{w}\mathscr{L}$ is a self-adjoint and positive semi-definite operator for any choice of $w > 0$, $p > 0$, $q \geq 0$.

b) *Find the eigenvalues and normalized eigenfunctions to L*

Now since L is a special case of $\mathscr{L}$, this is a Regular Sturm-Liouville problem

$$\frac{1}{w(x)}\mathscr{L}[u](x) = \lambda\, u(x)$$
$$-\mu\, u''(x) = \lambda\, u(x)$$

Hence $\lambda \geq 0$ by Property 2. If $\lambda > 0$, then the ODE is readily solvable to find the general solution of

$$u(x) = c_1 \cos\left(\sqrt{\frac{\lambda}{\mu}}\, x\right) + c_2 \sin\left(\sqrt{\frac{\lambda}{\mu}}\, x\right).$$

If we wish to apply the ICs $u(0) = 1$ and $u'(0) = 0$ we'd find that

$$u(0) = 1 \implies c_1 = 1$$
$$u'(0) = 0 \implies \sqrt{\lambda}\, c_2 = 0$$

Hence $v_1(x) = \cos\left(\sqrt{\frac{\lambda}{\mu}}\, x\right)$ and for the ICs $u(0) = 0$ and $u'(0) = 1$ we find that

$$u(0) = 0 \implies c_1 = 0$$
$$u'(0) = 1 \implies \sqrt{\frac{\lambda}{\mu}}\, c_2 = 1$$

And so $v_2(x) = \sqrt{\frac{\mu}{\lambda}} \sin\left(\sqrt{\frac{\lambda}{\mu}}\, x\right)$.
Hence by taking

$$u_k(x) = \beta_1 \cos\left(\sqrt{\frac{\lambda_k}{\mu}}\, x\right) + \alpha_1 \sqrt{\frac{\mu}{\lambda_k}} \sin\left(\sqrt{\frac{\lambda_k}{\mu}}\, x\right)$$

we know we satisfy the first BC (note I'm doing this as general as possible before applying the BCs. I do this because I recognize that my form of p, w, and q reduces this to something corresponding to the Heat equation. I often will want to solve the Heat equation with different BCs.). To satisfy the second BC we consider

$$u_k'(x) = -\beta_1 \sqrt{\frac{\lambda_k}{\mu}} \sin\left(\sqrt{\frac{\lambda_k}{\mu}}\, x\right) + \alpha_1 \cos\left(\sqrt{\frac{\lambda_k}{\mu}}\, x\right)$$

and so

$$\begin{aligned}
\alpha_2\, u(l) + \beta_2\, u'(l) &= 0\\
\alpha_2\,\beta_1\, v_1(l) + \alpha_2\,\alpha_1\, v_2(l) + \beta_2\,\beta_1\, v_1'(l) + \beta_2\,\alpha_1\, v_2'(l) &= 0\\
(\alpha_1\,\beta_2 + \alpha_2\,\beta_1)\,\cos\left(\sqrt{\frac{\lambda_k}{\mu}}\, l\right) + \left(\alpha_1\,\alpha_2\sqrt{\frac{\mu}{\lambda_k}} - \beta_1\,\beta_2\sqrt{\frac{\lambda_k}{\mu}}\right)\,\sin\left(\sqrt{\frac{\lambda_k}{\mu}}\, l\right) &= 0\\
(\alpha_1\,\beta_2 + \alpha_2\,\beta_1)\sqrt{\frac{\lambda_k}{\mu}}\,\cos\left(\sqrt{\frac{\lambda_k}{\mu}}\, l\right) + \left(\alpha_1\,\alpha_2 - \beta_1\,\beta_2\,\frac{\lambda_k}{\mu}\right)\,\sin\left(\sqrt{\frac{\lambda_k}{\mu}}\, l\right) &= 0
\end{aligned}$$

Now if we apply our particular BCs ($\alpha_1 = \alpha_2 = 1,\ \beta_1 = \beta_2 = 0$) we'll have

$$\sin\left(\sqrt{\frac{\lambda_k}{\mu}}\, l\right) = 0$$

$$\lambda_k = \mu\left(\frac{\pi k}{l}\right)^2, \quad k = 1, 2, \ldots$$

with eigenfunction

$$u_k(x) = \frac{l}{\pi k}\,\sin\left(\frac{\pi k}{l}\, x\right), \quad k = 1, 2, \cdots$$

To normalize these eigenfunctions we consider

$$\|u_k\| = \sqrt{\int_0^l (u_k(x))^2\, \mathrm{d}x} = \sqrt{\int_0^l \frac{l^2}{\pi^2 k^2}\,\sin^2\left(\frac{\pi k}{l}\, x\right)\mathrm{d}x} = \sqrt{\frac{1}{2}\frac{l^3}{\pi^2\, k^2}} = \frac{1}{\sqrt{2}}\frac{l^{3/2}}{\pi k}$$

hence the normalized version of u_k is defined as

$$\hat{u}_k = \frac{u_k}{\|u_k\|} = \sqrt{\frac{2}{l}}\,\sin\left(\frac{\pi k}{l} x\right), \quad k = 1, 2, \ldots.$$

From Property 2 we know that we don't need to check $\lambda < 0$, but we *do* need to check $\lambda = 0$ still. In which case we have,

$$-\mu\, u'' = 0$$

with general solution

$$u(x) = c_1\, x + c_2$$

hence applying the $u(0) = 1,\ u'(0) = 0$ ICs we have

$$v_1(x) = 1$$

and applying the $u(0) = 0,\ u'(0) = 1$ ICs we have

$$v_2(x) = x$$

hence

$$u_0(x) = \beta_1 + \alpha_1\, x$$

to apply the BC we note that

$$\begin{aligned}
\alpha_2\, v_1(l) + \beta_2\, v_2(l) &= 0\\
\alpha_2\,\beta_1\, v_1(l) + \alpha_2\,\alpha_1\, v_2(l) + \beta_2\,\beta_1\, v_1'(l) + \beta_2\,\alpha_1\, v_2'(l) &= 0\\
\alpha_2\,\beta_1 + \alpha_2\,\alpha_1\, l + \beta_2\,\alpha_1 &= 0
\end{aligned}$$

Now this equation contains no λ dependence. So we cannot control it to set everything equal to zero. In particular, for a regular Sturm-Liouville problem this is true if and only if $\alpha_1 = \alpha_2 = 0$. In our particular problem, $\alpha_1 = \alpha_2 = 1 \neq 0$, hence $\lambda = 0$ is not an eigenvalue of our particular Sturm-Liouville problem.

Thus the eigenvalues and normalized eigenfunctions for this problem are

$$\lambda_k = \mu \left(\frac{\pi k}{l}\right)^2, \quad \hat{u}_k = \sqrt{\frac{2}{l}} \sin\left(\frac{\pi k}{l} x\right), \quad k = 1, 2, \ldots$$

Example 2. *Solve the following Heat equation via separation of variables*

$$u_t(x,t) = D\, u_{xx}(x,t), \quad u(x,0) = f(x), \quad u(0) = 0, \quad u(2) = 0, \quad 0 \le x \le 2 \tag{1}$$

where

$$f(x) = \begin{cases} -1 & 0 \le x \le 1 \\ 1 & 1 < x \le 2 \end{cases}.$$

(This is the same example as in Lec 12 with l and $f(x)$ specified). We saw already that taking $u(x,t) = X(x)\, T(t)$ gives

$$\begin{aligned} \frac{T'(t)}{T(t)} &= D \frac{X''(x)}{X(x)} \\ -\frac{T'(t)}{T(t)} &= \lambda = -D \frac{X''(x)}{X(x)}. \end{aligned}$$

Now that last step might seem weird, but if we do so then we ensure that the eigenvalue problem

$$-D\, X''(x) = \lambda\, X(x), \quad X(0) = 0 = X(2)$$

is a Regular Sturm-Liouville type: which is the same as Example 1 with $\mu = D$, hence

$$\lambda_k = D \left(\frac{\pi\, k}{2}\right)^2, \quad \hat{X}_k = \sin\left(\frac{\pi\, k}{2}\, x\right).$$

We saw last time that we need to solve the T ODE with the IC $T(0) = (f, \hat{X}_k)$ and

$$\begin{aligned} (f, \hat{X}_k) &= \int_0^2 w(x)\, f(x)\, \hat{X}_k \, \mathrm{d}x \\ &= \int_0^1 -\hat{X}_k \, \mathrm{d}x + \int_1^2 \hat{X}_k \, \mathrm{d}x \\ &= \int_0^1 -\sin\left(\frac{\pi\, k}{2}\, x\right) \mathrm{d}x + \int_1^2 \sin\left(\frac{\pi\, k}{2}\, x\right) \mathrm{d}x \\ &= \frac{-2\, \cos(\pi\, k) + 4\, \cos\left(\frac{1}{2}\, \pi\, k\right) - 2}{\pi\, k} \end{aligned}$$

If k is odd then $\cos(\pi\, k) = -1$ and $\cos\left(\frac{1}{2}\, \pi\, k\right) = 0$, hence $(f, \hat{X}_k) = 0$ for odd k. If $k = 2\, n$ is even, then $\cos(2\, \pi\, n) = 1$ and $\cos(\pi\, n) = (-1)^n$ hence

$$(f, \hat{X}_k) = \frac{2\, (-1)^n - 2}{\pi\, n} = \begin{cases} 0 & n \text{ even} \\ -\frac{4}{\pi\, n} & n \text{ odd} \end{cases}$$

Now that we have the values of $(f, \hat{X}_k)$ we can solve the T ODEs and construct our solution.