1 Example 2, Continued

Last time we wanted to solve the heat equation

$$u_t(x,t) = D\,u_{xx}(x,t), \quad u(x,0) = f(x), \quad u(0,t) = 0, \quad u(2,t) = 0, \quad 0 \le x \le 2 \tag{1}$$

where

$$f(x) = \begin{cases} -1 & 0 \le x \le 1 \\ 1 & 1 < x \le 2 \end{cases}$$

via separation of variables.
We separated the equation and found that for positive integers n,

$$(f, \hat{X}_{2n-1}) = 0, \quad (f, \hat{X}_{2n}) = \frac{2\,(-1)^n - 2}{\pi\, n} = \begin{cases} 0 & n \text{ even} \\ -\frac{4}{\pi\, n} & n \text{ odd} \end{cases}$$

$$\{(f, \hat{X}_k)\} = \left\{0, -\frac{4}{\pi}, 0, 0, 0, -\frac{4}{3\,\pi}, 0, 0, 0, -\frac{4}{5\,\pi}, \ldots\right\}$$

so the non-zero coefficients correspond to $k = 2\,(2\,j-1) = 4\,j-2$ for $j = 1, 2, \ldots$.
Now the T ODE

$$-T'(t) = \lambda_k\, T(t), \quad T(0) = (f, \hat{X}_k)$$

has solution $T_k(t) = (f, \hat{X}_k)\exp(-\lambda_k\, t)$ and so $T = 0$ when $(f, \hat{X}_k) = 0$.
Therefore $u_k(x,t) = T_k(t)\, \hat{X}_k(x)$ and so

$$\begin{aligned} u(x,t) = \sum_{k=1}^{\infty} u_k(x,t) &= \sum_{k=1}^{\infty} T_k(t)\, \hat{X}_k(x) \\ &= \sum_{j=1}^{\infty} T_{4j-2}(t)\, \hat{X}_{4j-2}(x) \\ &= \sum_{j=1}^{\infty} -\frac{4}{\pi\,(2j-1)}\, e^{-D\,\pi^2\,(2j-1)^2\,t}\, \sin\left(\pi\,(2\,j-1)\,x\right). \end{aligned}$$

Note: while finding u we found the generalized Fourier series of $f(x)$ in terms of our eigenfunctions $\hat{X}_k$. For the particular eigenfunctions, this corresponds to a Fourier sine series. We know that this series converges due to our properties of Sturm-Liouville problems due to the fact that f is square integrable. This expansion of $f(x)$ gives rise to what is known as Gibb's phenomenon - a common phenomenon in the Fourier series of a discontinuous function (`See Maple worksheet`).

Definition 1 (Gibb's Phenomenon (summary))**.** *If $f(x)$ has a discontinuity, then the partial sums of the Fourier series of f will over-shoot and under-shoot the true value of $f(x)$ near the discontinuity. This overshoot approaches a constant as $N \to \infty$ and the overshoot is roughly 9% of the jump height.*

We know that the series

$$\sum_{k=1}^{\infty} (f, \hat{X}_k)\, \hat{X}(x)$$

converges to f in a mean squared sense. However, the series always has *some* error (due to Gibb's phenomenon). There are many other methods by which we could discuss converegence of a series:

Definition 2. *Consider the partial sums*

$$S_n = \sum_{i=0}^{n} f_n(\vec{x})$$

then

1. *The sequence* $\{S_n\}$ *converges pointwise to* $f(\vec{x})$ *on a domain* D *if*

   $$\lim_{n\to\infty} S_n(\vec{x}) = f(\vec{x})$$

   for all $\vec{x} \in D$

2. *The sequence* $\{S_n\}$ *converges uniformly to* $f(x)$ *on a domain* D *if*

   $$\lim_{n\to\infty} \sup_{\vec{x}\in D} |S_n(\vec{x}) - f(\vec{x})| = 0$$

If the partial sums converge pointwise/uniformly to f*, then we say the series converges pointwise/uniformly to* f*.*

Intuitively pointwise convergence means: if I sit at any particular x point, the series will become arbitrarily close to that value. Uniform convergence means: the maximum error of my partial sums tends to zero **or, alternatively** the partial sums become arbitrarily good approximators of f over the whole domain. In the Gibb's phenomenon worksheet, we saw that the maximum error for any partial sum was always a positive number, but for each progressive n this error was pushed closer and closer to the discontinuities. In a pointwise sense, the series converges; in a uniform sense, the series diverges.