1 Termwise Differentiation

From the definition it is fairly straightforward to see that uniform convergence implies pointwise convergence. Uniform convergence also implies mean-squared convergence. To see this, suppose $\{f_n\} \to f$ uniformly on $[0, l]$. As a result of this, given any $\epsilon > 0$ we can find an N such that for $n \geq N$

$$|f_n(x) - f(x)| < \frac{\epsilon}{\sqrt{l}}, \quad \forall\, 0 \leq x \leq l.$$

Hence for unit weight with $n \geq N$ we have

$$\|f_n - f\|^2 = \int_0^l |f_n(x) - f(x)|^2 \,\mathrm{d}x < \int_0^l \frac{\epsilon^2}{l} \,\mathrm{d}x = \epsilon^2$$

thus uniform convergence implies mean squared convergence as well! Hence uniform convergence is a stronger measure of convergence. There are lots of properties of uniform convergence that are desirable when compared to pointwise convergence. For instance, on the domain $[0, 1]$ the sequence of functions $f_n(x) = x^n$ converges pointwise to the function

$$f(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}$$

For $x = 1$ we have $f_n(1) = f(1)$, however for $0 \leq x < 1$ we have

$$|f_n(x) - f(x)| = x^n$$

which, for fixed n, can be made arbitrarily large by taking x arbitrarily close to 1. That is,

$$\sup_{0 \leq x \leq 1} |f_n(x) - f(x)| = \sup_{0 \leq x < 1} x^n = 1$$

and so f_n does *not* converge uniformly to f. Proving this required us to know f, which we often don't know explicitly. So using the definition to prove uniformity can be unwieldly, thankfully we usually use a shortcut:

Theorem 1 (Weierstrass M-test)**.** *If $S_n = \sum_{k=1}^n f_k$ is a sequence of functions on $[0, l]$ sastisfying*

$$\sup_{x \in [0,l]} |S_n(x) - S_{n-1}(x)| = \sup_{x \in [0,l]} |f_n(x)| \leq M_n$$

for some particular M_n where $\sum_{n=1}^\infty M_n < \infty$, then S_n converges uniformly on $[0, l]$ (equivalently, the series $\sum_{n=1}^\infty f_n(x)$ converges uniformly on $[0, l]$).

With the example of $\{x^n\}$ we saw that the pointwise limit of continuous functions is not necessarily continuous. As a corollary to this, the pointwise limit of differentiable functions is not necessarily differentiable. However,

Theorem 2. *If*

$$\sum_{n=1}^\infty f_n(x) = f(x)$$

uniformly on the interval $[0, l]$ and f_n is continuous on $[0, l]$ for each n, then $f(x)$ is continuous on $[0, l]$ and

$$\sum_{n=1}^\infty \int_0^l f_n(x) \,\mathrm{d}x = \int_0^l f(x) \,\mathrm{d}x.$$

(i.e. the uniform limit of continuous functions is continuous and uniformly convergent series can be integrated termwise).

Theorem 3. *If* $\sum_{n=1}^{\infty} f_n(c)$ *is convergent for any one* $c \in [0,l]$ *and* $\sum_{n=1}^{\infty} f_n'(x)$ *converges uniformly on* $[0,l]$*, then* $\sum_{n=1}^{\infty} f_n(x)$ *converges uniformly on* $[0,l]$ *to some function* $f(x)$ *and*

$$\sum_{n=1}^{\infty} f_n'(x)\,\mathrm{d}x = f'(x)$$

(i.e. uniformly convergent series can be differentiated termwise).

Example 1.

For a counter-example to term-by-term differentiation, consider the function $f(x) = x$ on $x \in [0,l]$ with $\hat{X}_k = \sqrt{\frac{2}{l}}\,\sin\left(\frac{k\,\pi}{l}x\right)$. The generalized Fourier series is then

$$x = \sum_{k=1}^{\infty} (f, \hat{X}_k)\,\hat{X}_k(x) = \sum_{k=1}^{\infty} \frac{2\,l}{k\,\pi}\,(-1)^{k+1}\,\sin\left(\frac{k\,\pi}{l}x\right), \quad x \in [0,l]$$

and if we differentiate the series termwise we obtain

$$\sum_{k=1}^{\infty} 2\,(-1)^{k+1}\,\cos\left(\frac{k\,\pi}{l}x\right). \tag{1}$$

Now this result is weird for many reasons. First of all, (1) clearly isn't in terms of $\hat{X}_k$ anymore. That's fine, if we define

$$\hat{Y}_k(x) = \sqrt{\frac{2}{l}}\,\cos\left(\frac{k\,\pi}{l}\,x\right)$$

and consider the generalized Fourier series of f' in terms of $\hat{Y}_k$ (in this case, a Fourier cosine series of f') we would get something completely different than (1). Moreover, recall (from Math 138 or similar) that

$$\sum_{k=1}^{\infty} a_k$$

converges only if $\lim_{k\to\infty} a_k = 0$ and

$$\lim_{k\to\infty} 2\,(-1)^{k+1}\,\cos\left(\frac{k\,\pi}{l}x\right)$$

does not even converge in general (let alone decay to zero). So in this case termwise differentiation gave us something that wasn't even convergent let alone equal to the general Fourier series of the derivative function ($f'(x) = 1$).
So back to our heat equation separation of variables, are we justified in taking derivatives termwise? The answer is **no**, in general. Near the discontinuities Gibb's phenomenon will always appear, hence we are not able to achieve uniform convergence. Moreover we can show that our separation of variables solution is not correct at the discontinuities. For instance, $u_k(1,t) = 0 \quad \forall k$, but the *real* solution of the PDE is non-zero at $x = 1$ (initially, $u(1,0) = -1$ and it will exponentially decay to zero as time increases). We saw that the generalized Fourier series of the discontinuous $f(x)$ converged pointwise on $(0,1) \cup (1,2)$. In fact for any $\epsilon > 0$, the Fourier series converges *uniformly* on

$$[\epsilon, 1-\epsilon] \cup [1+\epsilon, 2-\epsilon]$$

because the over-shoot/under-shoot from Gibb's phenomenon will be pushed into that ϵ-buffer around the discontinuities by taking more terms in the series. So our problems here were because $f(x)$ was discontinuous, but the separation of variables solution *is* trustworthy on a domain that is separated from the discontinuities.