1 Examples

Example 1. *Prove that the following series*

$$\sum_{k=1}^{\infty} f_k(x) = \sum_{k=1}^{\infty} \frac{k^2 + x^4}{k^4 + x^2} \tag{1}$$

converges to a function that is continuous for all $x \in \mathbb{R}$

This proof will continue in two parts.

1. We will use Weierstrass' M-Test to show that the series converges uniformly on $[-l, l]$ for some $l > 0$
2. We will then conclude that the function must be continuous on all $\mathbb{R}$

Let $S_n(x)$ denote the nth partial sum of the series in (1) and let $l > 0$ be given. The M-test says that if

$$\sup_{x\in[0,l]} |S_n(x) - S_{n-1}(x)| = \sup_{x\in[0,l]} \left| \frac{n^2 + x^4}{n^4 + x^2} \right| \leq M_n$$

where the M_ns form a convergent series, then the series in (1) converges uniformly on $[0, l]$. Now for $x \in [0, l]$ we have

$$\begin{aligned} \left| \frac{n^2 + x^4}{n^4 + x^2} \right| &\leq \frac{n^2 + x^4}{n^4} \\ &\leq \frac{1}{n^2} + \frac{l^4}{n^4} = M_n \end{aligned}$$

The series

$$\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^2} + l^4 \sum_{n=1}^{\infty} \frac{1}{n^4}$$

is convergent (two p-series with $p > 1$). Hence (1) converges uniformly on $[0, l]$. Now note that $S_n(x) = S_n(-x)$, hence (1) converges uniformly on $[-l, l]$. (Alternatively, change units in x so that $\hat{x} \in [-l, l]$).
First note that $f_k(x)$ is continuous for any k or x. Now for any $x \in \mathbb{R}$, we can find an l such that $x \in [-l, l]$. Then, since (1) converges uniformly on this restricted interval, it must be that (1) is continuous at x. Since the choice of x is arbitrary, we have that (1) is continuous on all $\mathbb{R}$.

Example 2 (Example 1b.)**.** *Is it the case that* (1) *is uniformly convergent on* $\mathbb{R}$*?*

No. The argument above is enough to conclude pointwise convergence of (1). Hence let $f(x) = \sum_{k=1}^{\infty} f_k(x)$, To be uniformly convergent on $\mathbb{R}$ the supremum would be taken over $\mathbb{R}$, that is

$$\sup_{x\in\mathbb{R}} |S_n(x) - f(x)| = \sup_{x\in\mathbb{R}} \left| \sum_{k=n+1}^{\infty} \frac{n^2 + x^4}{n^4 + x^2} \right| = \infty$$

so the series does *not* converge uniformly.

2 Relation to Generalized Fourier Series

Recall our general boundary conditions for a domain V are

$$\alpha(\vec{x})\, u + \beta(\vec{x})\, \frac{\partial u}{\partial n}\Big|_{\partial V} = B(\vec{x}, t)$$

for $x \in \partial V$ (so-called Robin BCs). Another type of BC we considered at the beginning of the course was the periodic boundary condition. In 1D these are BCs like

$$u(0,t) = u(l,t), \quad u_x(0,t) = u_x(l,t)$$

Theorem 1. *If f is continuous with continuous first and second derivatives on $[0,l]$ (i.e. $f \in C^2([0,l])$) and f satisfies Robin or periodic boundary conditions, then the generalized Fourier series of f converges uniformly on $[0,l]$.*

So in the context of using eigenfunctions to solve PDEs, this means that as long as our initial data is C^2, the separation of variables technique is trustworthy *everywhere*. Contrast this with last week, where discontinuous initial data led us to solutions that were only trustworthy *away* from the discontinuities.