Example 1. *Solve the 2D wave equation (in polar coordinates) for a vibrating drum head of radius 1 via separation of variables.*

The PDE becomes

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2_{\text{polar}} u = c^2 \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right]$$

on the domain $0 \leq \theta \leq 2\pi$ with $0 < r < 1$.
For boundary conditions we consider

$$\begin{aligned} u(1, \theta, t) &= 0 \\ u(r, \theta, 0) &= 1 - r^2 \\ u_t(r, \theta, 0) &= 0 \\ u(r, 0, t) &= u(r, 2\pi, t) \\ u_\theta(r, 0, t) &= u_\theta(r, 2\pi, t) \end{aligned}$$

and

$$u(0, \theta, t) \text{ is finite}$$

Using separation of variables with $u(r, \theta, t) = R(r)\, \Theta(\theta)\, T(t)$, then

$$\begin{aligned} R\, \Theta\, T'' &= c^2 \left(\frac{\Theta\, T}{r} \frac{\partial}{\partial r} \left(r\, R' \right) + \frac{1}{r^2} R\, \Theta''\, T \right) \\ \frac{T''}{T} &= c^2 \left(\frac{1}{r\, R} \frac{\partial}{\partial r} \left(r\, R' \right) + \frac{1}{r^2} \frac{\Theta''}{\Theta} \right) \end{aligned}$$

as before, the LHS is dependent on t and the RHS on r and θ, hence we introduce a separation constant $-\lambda$ so that

$$\frac{T''}{T} = -\lambda = c^2 \left(\frac{1}{r\, R} \frac{\partial}{\partial r} \left(r\, R' \right) + \frac{1}{r^2} \frac{\Theta''}{\Theta} \right)$$

Hence temporally we have the ODE

$$T'' + \lambda\, T = 0$$

for the RHS note that

$$\begin{aligned} -\lambda &= c^2 \left(\frac{1}{r\, R} \frac{\partial}{\partial r} \left(r\, R' \right) + \frac{1}{r^2} \frac{\Theta''}{\Theta} \right) \\ -r^2 \lambda &= \frac{c^2\, r}{R} \frac{\partial}{\partial r} \left(r\, R' \right) + c^2 \frac{\Theta''}{\Theta} \\ -c^2 \frac{\Theta''}{\Theta} &= \frac{r\, c^2}{R} \frac{\partial}{\partial r} (r\, R') + \lambda\, r^2 \end{aligned}$$

where this equation has functional dependence cancelling as well. So I introduce a *second* separation constant μ so that

$$-c^2\, \Theta'' = \mu\, \Theta$$

which has the dynamic form of a Sturm-Liouville problem with $p(\theta) = c^2 > 0$, $q(\theta) = 0 \geq 0$, and $w(\theta) = 1 > 0$ (we'll worry about BCs next time). Similarly for R we have

$$\begin{aligned} c^2\, (r\, R')' + \lambda\, r\, R &= \frac{\mu}{r} R \\ \frac{r}{c^2} \lambda\, R &= -(r\, R')' + \frac{\mu}{r\, c^2} R \end{aligned}$$

which is (ignoring BCs for now) *also* of Sturm-Liouville type with $p(r) = r > 0$, $q(r) = \mu/c^2 \geq 0*$, and $w(r) = r/c^2 > 0$. (The asterisk is because we need to justify the non-negativity of μ – we're not guaranteed that this eigenvalue is non-negative because it's the eigenvalue for our eigenfunctions Θ: the Θ BCs are not included in our Sturm-Liouville theory, so we'll need to justify the sign of μ manually). Now let's focus on the BCs. If we translate the BCs from u to our separated functions we have:

$$R(1) = 0, \quad T'(0) = 0, \quad \Theta(0) = \Theta(2\pi), \quad \Theta'(0) = \Theta'(2\pi), \quad R(0) = R_0 \quad \text{for some } R_0 \in \mathbb{R}$$

Let's focus on the Θ equation first. Since we do not *a-priori* know the sign of μ we should proceed casewise. Sparing the details, $\mu \leq 0$ results in the trivial solution when we try to impose the periodic boundary conditions. Hence we assume $\mu > 0$ in which case we get

$$\Theta(\theta) = A\cos\left(\frac{\sqrt{\mu}}{c}\,\theta\right) + B\sin\left(\frac{\sqrt{\mu}}{c}\,\theta\right)$$

imposing the periodic boundary conditions yields either $A = B = 0$ or $\mu = (c\,n)^2$ where $n = 0, 1, 2, \ldots$. Hence,

$$\Theta_n(\theta) = A_n \cos(n\,\theta) + B_n \sin(n\,\theta).$$

`There's a lot of tedious work I'm sweeping under the rug here (the non-positive` μ`s, that` $\mu = (c\,n)^2$ `is the` ***only*** `non-trivial solution, etc). None of this work is` ***deep***`, it's just straight forward (but time-consuming) calculation.`

Now let's focus on the R equation which, under this solution for μ, gives us

$$-(r\,R')' + \frac{n^2}{r}\,R = \frac{r}{c^2}\,\lambda\,R, \quad R(0) = R_0, \quad R(1) = 0$$

which is a Sturm-Liouville problem (even the BCs!). So we know $\lambda \geq 0$. Actually this ODE is a famous one, it is the Bessel equation of order n with general solution

$$R(r) = D\,J_n\left(\frac{\sqrt{\lambda}}{c}\,r\right) + E\,Y_n\left(\frac{\sqrt{\lambda}}{c}\,r\right)$$

where J_k and Y_k are the kth order Bessel functions of first and second kind (respectively). If we apply the BCs we have

$$R(0) = D\,J_n(0) + \lim_{r\to 0^+} E\,Y_n\left(\frac{\sqrt{\lambda}}{c}\,r\right)$$

it can be shown that

$$\lim_{r\to 0^+} Y_n(\sqrt{\lambda_m}\,r) = -\infty$$

and so we must have the $E = 0$ (as an aside, $J_n(0) = \delta_{n,0}$ where $\delta_{i,j}$ is Kronecker's delta). Hence

$$R(r) = D\,J_n\left(\frac{\sqrt{\lambda}}{c}\,r\right).$$