Drum Example Continued

Example 1. *Solve the 2D wave equation (in polar coordinates) for a vibrating drum head of radius 1 via separation of variables.*

Last time we applied the $r = 0$ BC to find that

$$R(r) = D\, J_n\left(\frac{\sqrt{\lambda}}{c}\, r\right).$$

If we applying the other BC at $r = 1$ we see that either $D = 0$ (which we reject) or

$$J_n\left(\frac{\sqrt{\lambda}}{c}\right) = 0. \tag{1}$$

Since the R ODE was a Sturm-Liouville problem it must be the case that we can find countably infinitely many non-negative λs that make (1) true. This is part of the power of Sturm-Liouville analysis: the Bessel functions are very complicated, and the order of these Bessel functions is not necessarily integer, however these λs exist. To find these roots in practice, one would either use a numerical solver or consult a table of values. In any case, we label these roots $\sqrt{\lambda_{m,n}}/c$ for $m = 1, 2, \ldots$. (Here the n indicates (part) of the order the Bessel function. That Bessel function of order n has infinitely many non-negative zeros: which we count by the index m: that is n comes from the μ eigenvalue and m from the λ eigenvalue).
All together,

$$R_m(r) = D_m\, J_n\left(\frac{\sqrt{\lambda_{m,n}}}{c}\, r\right).$$

Now finally, for the T equation we have

$$T'' + \lambda_{m,n}\, T = 0$$

where we know that $\lambda_{m,n} \geq 0$. Hence,

$$T(t) = F\, \sin\left(\sqrt{\lambda_{m,n}}\, t\right) + G\, \cos\left(\sqrt{\lambda_{m,n}}\, t\right)$$

but since $T'(0) = 0$ we have $F = 0$.
Then

$$u_{m,n}(r, \theta, t) = (a_{m,n}\, \cos(n\,\theta) + b_{m,n}\, \sin(n\,\theta))\; J_n\left(\frac{\sqrt{\lambda_{m,n}}}{c}\, r\right)\, \cos\left(\sqrt{\lambda_{m,n}}\, t\right)$$

All together our solution becomes

$$u(r, \theta, t) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} u_{m,n}(r, \theta, t)$$

where we've applied all BCs/ICs *except* the $u(r, \theta, 0) = 1 - r^2$ IC.
Now the functions $u_{m,n}(r, \theta, t)$ are called the *modes* or the *harmonics* of the system. The zeroes of the modes are called the *nodes* and the places where the modes reach maximum amplitudes are called *antinodes*. Often you can recover much of the spatial structure of the full solution by plotting only a few modes. While the modes technically depend upon t, often we fix t to a particular value and only consider the spatial structure of the modes.

To apply the final IC we consider

$$u(r,\theta,0) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} (a_{m,n}\,\cos(n\,\theta) + b_{m,n}\,\sin(n\,\theta))\;J_n\left(\frac{\sqrt{\lambda_{m,n}}}{c}\,r\right) = 1 - r^2.$$

but first I'll define

$$\hat{a}_{m,n} = a_{m,n}\left\| J_n\left(\frac{\sqrt{\lambda_{m,n}}}{c}\,r\right)\right\|$$

and $\hat{b}_{m,n}$ analogously so that the $u(r,\theta,0)$ equation becomes

$$u(r,\theta,0) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \left(\hat{a}_{m,n}\,\cos(n\,\theta) + \hat{b}_{m,n}\,\sin(n\,\theta)\right)\;\hat{J}_n\left(\frac{\sqrt{\lambda_{m,n}}}{c}\,r\right) = 1 - r^2.$$

where the Bessel functions have now been normalized.
We want to find the values of $\hat{a}_{m,n}$ and $\hat{b}_{m,n}$. First, take it as fact that for $k \neq 0$ with integer k and m that

$$\int_0^{2\,\pi} \cos(k\,\theta)\,\cos(m\,\theta)\,\mathrm{d}\theta = \pi\,\delta_{m,k}$$
$$\int_0^{2\,\pi} \sin(k\,\theta)\,\sin(m\,\theta)\,\mathrm{d}\theta = \pi\,\delta_{m,k}$$
$$\int_0^{2\,\pi} \sin(k\,\theta)\,\cos(m\,\theta)\,\mathrm{d}\theta = 0.$$

(These are standard calculations often first justified in an intro to Fourier series course. The process of obtaining these relations is a straightforward exercise in first year calculus).
Hence for $k \neq 0$

$$\int_0^{2\pi} \cos(k\,\theta)\,u(r,\theta,0)\,\mathrm{d}\theta = \int_0^{2\pi} \cos(k\,\theta)\,(1-r^2)\,\mathrm{d}\theta$$

or, substituting in our series form, (in this next step I'm justified in exchanging the order of the integral and the series operator because of a theorem from last Wednesday (Lec 21): namely, since $1-r^2$ is a C^2 function in θ on $[0, 2\,\pi]$ and Θ and f satisfy the periodic boundary conditions we have that the series converges uniformly. The Bessel function factor can be viewed as a constant, as far as θ is concerned).

$$\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\hat{J}_n\left(\frac{\sqrt{\lambda_{m,n}}}{c}\,r\right)\left(\hat{a}_{m,n}\int_0^{2\pi}\cos(n\,\theta)\cos(k\,\theta)\,\mathrm{d}\theta + \hat{b}_{m,n}\int_0^{2\pi}\sin(n\,\theta)\cos(k\,\theta)\,\mathrm{d}\theta\right) = 0$$
$$\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\hat{J}_n\left(\frac{\sqrt{\lambda_{m,n}}}{c}\,r\right)\,\hat{a}_{m,n}\,\delta_{n,k} = 0$$
$$\sum_{m=1}^{\infty}\hat{J}_k\left(\frac{\sqrt{\lambda_{m,k}}}{c}\,r\right)\,\hat{a}_{m,k} = 0$$

Hence $\hat{a}_{m,k} = \left(0, \hat{J}_k\left(\frac{\sqrt{\lambda_{m,k}}}{c}\,r\right)\right) = 0$ if $k \neq 0$. A very similar argument shows $\hat{b}_{m,k} = 0$ if $k \neq 0$. Now note that the value $\hat{b}_{m,0}$ doesn't affect the form of u since $\sin(n\,\theta) = 0$ for $n = 0$ and so we take $\hat{b}_{m,0} = 0$. All that to say,

$$u(r,\theta,0) = \sum_{m=1}^{\infty}\hat{a}_{m,0}\,\hat{J}_0\left(\frac{\sqrt{\lambda_{m,0}}}{c}\,r\right) = 1 - r^2$$

and so

$$\hat{a}_{m,0} = \left(1 - r^2, \hat{J}_0\left(\frac{\sqrt{\lambda_{m,0}}}{c}\,r\right)\right).$$

Note this argument works for more than just $1 - r^2$ – it works for any IC of the form $u(r, \theta, 0) = f(r)$ by replacing $1 - r^2$ with $f(r)$ above. All told our solution to the PDE is then

$$u(r, \theta, t) = \sum_{m=1}^{\infty} \hat{a}_{m,0} \, \hat{J}_0 \left(\frac{\sqrt{\lambda_{m,0}}}{c} \, r \right) \cos\left(\sqrt{\lambda_{m,0}} \, t \right)$$

1. The solution u is independent of θ *because* my initial data were independent of θ.
2. Through a series of calculations by taking $w(r) = r$ in the inner product we *could* find the values of $\hat{a}_{m,0}$ and the normalization constants (or equivalently, the values of $a_{m,0}$) in terms of the values of *other* Bessel functions J_1 and J_2 evaluated at $\sqrt{\lambda_{m,0}}/c$ (the m roots of J_0). This isn't necessarily *practical* as the values of the roots of J_0 are things we need a computer for anyway (but it *is* cool).
3. If t is measured in seconds, then $\sqrt{\lambda_{m,0}}$ is measured in Hz (due to the cos factor). These λs intimately depend upon our boundary conditions. In a really (really) cool paper, it was shown that you can go backwards: given the λs, you can determine the shape of the boundary (these inverse-eigenvalue problems are very important in physics/biology). Now these $\sqrt{\lambda}$s are measured in Hz and correspond to the tones of the drum. All that to say, if you listen *very very* closely, you can hear the shape of a drum.