Duhamel's Principle

So far we have only dealt with homogeneous PDEs, now we turn our attention to inhomogeneous PDEs and what is known as Duhamel's principle. For our purposes this will allow us to translate inhomogeneous PDEs (or forced PDEs) into homogeneous ones. To begin, recall the hyperbolic and parabolic equations (Lec 12)

$$\underbrace{w(x)\,u_{tt}(x,t) + \mathscr{L}[u](x,t) = w(x)\,F(x,t),\ \ t>0}_{\text{hyperbolic}} \quad \text{and} \quad \underbrace{w(x)\,u_t(x,t) + \mathscr{L}[u](x,t) = w(x)\,F(x,t),\ \ t>0}_{\text{parabolic}}$$

(for non-zero F this is inhomogeneous). While we're focused on the 1D problem, the method works for any number of spatial dimensions. For the moment we'll weaken our initial conditions to the homogeneous ICs:

$$\underbrace{u(x,0)=0,\ u_t(x,0)=0}_{\text{hyperbolic}} \quad \text{and} \quad \underbrace{u(x,0)=0}_{\text{parabolic}}.$$

This process works for both infinite spatial domains and finite spatial domains. In order to use Duhamel's principle, we *instead* consider the related homogeneous problem in terms of a new function $v(x,t;\tau)$

$$\underbrace{w\,v_{tt} + \mathscr{L}[v] = 0,\ t>\tau}_{\text{hyperbolic}} \quad \text{and} \quad \underbrace{w\,v_t + \mathscr{L}[v] = 0,\ t>\tau}_{\text{parabolic}}$$

where v has different initial conditions. First of all, the ICs take place at $t=\tau$ and are

$$\underbrace{v(x,\tau;\tau)=0,\ v_t(x,\tau;\tau)=F(x,\tau)}_{\text{hyperbolic}} \quad \text{and} \quad \underbrace{v(x,\tau;\tau)=F(x,\tau)}_{\text{parabolic}}.$$

This translated PDE in terms of v is something that we know how to solve using methods from earlier in the course (characteristics/separation). If we're dealing with a finite spatial domain, then v satisfies the same homogeneous BCs. Duhamel's principle then states that

$$u(x,t) = \int_0^t v(x,t;\tau)\,\mathrm{d}\tau.$$

To justify this method we need to be able to calculate the values of u_t and u_{tt}, for which we will need to utilize **Leibniz's Rule**

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{b(t)} f(x,t)\,\mathrm{d}x = b'(t)\,f(b(t),t) - a'(t)\,f(a(t)\,t) + \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}(x,t)\,\mathrm{d}x$$

Hence,

$$\begin{aligned} u_t(x,t) &= \frac{\partial}{\partial t}\left(\int_0^t v(x,t;\tau)\,\mathrm{d}\tau\right) \\ &= v(x,t;t) + \int_0^t v_t(x,t;\tau)\,\mathrm{d}\tau \\ u_{tt}(x,t) &= \frac{\partial}{\partial t}\left(v(x,t;t)\right) + v_t(x,t;t) + \int_0^t v_{tt}(x,t;\tau)\,\mathrm{d}\tau \end{aligned}$$

We note that in both the parabolic and hyperbolic case the IC $u(x,0) = 0$ is satisfied (since $u(x,0) = \int_0^0 v(x,t;\tau)\,d\tau = 0$). Now we'll justify the parabolic case. In the parabolic case we have

$$
\begin{aligned}
w(x)\,u_t + \mathscr{L}[u] &= w(x)\,\left(v(x,t;t) + \int_0^t v_t(x,t;\tau)\,\mathrm{d}\tau\right) + \mathscr{L}\left[\int_0^t v(x,t;\tau)\,\mathrm{d}\tau\right] \\
&= w(x)\,F(x,t) + w(x)\,\int_0^t v_t(x,t;\tau)\,\mathrm{d}\tau + \int_0^t \mathscr{L}\left[v(x,t;\tau)\right]\,\mathrm{d}\tau \\
&= w(x)\,F(x,t) + \int_0^t w\,v_t + \mathscr{L}\left[v\right]\,\mathrm{d}\tau \\
&= w(x)\,F(x,t)
\end{aligned}
$$

Moving from line (1) to (2) above we are using the $v(x,\tau;\tau)$ initial condition with $\tau \mapsto t$ and we're bringing $\mathscr{L}$ under the integral since $\mathscr{L}$ only deals with spatial derivatives. Finally, the last step from lines (3) to (4) is justified because v solves the PDE for $t > \tau$. Since we're integrating τ between 0 and t we have $\tau < t$. Since the only IC for the parabolic case is satisfied, u solves the parabolic IVP.
Very similarly in the hyperbolic case we have

$$
\begin{aligned}
w(x)\,u_{tt} + \mathscr{L}[u] &= w(x)\,\left(\frac{\partial}{\partial t}(0) + F(x,t) + \int_0^t v_{tt}(x,t;\tau)\,\mathrm{d}\tau\right) + \mathscr{L}\left[\int_0^t v(x,t;\tau)\,\mathrm{d}\tau\right] \\
&= w(x)\,F(x,t) + \int_0^t w\,v_{tt} + \mathscr{L}[v]\,\mathrm{d}\tau \\
&= w(x)\,F(x,t).
\end{aligned}
$$

In this case we have the additional IC $u_t(x,0) = 0$ and

$$
u_t(x,0) = v(x,0;0) + \int_0^0 v_t(x,t;\tau)\,\mathrm{d}\tau = 0.
$$

(Where the first term is zero because of the IC for v in the hyperbolic case).
This whole process depends upon us having an inhomogeneous PDE for u with homogeneous ICs. In full generality if we wanted to solve a PDE like

$$
w\,u_{tt} + \mathscr{L}[u] = w(x)\,F(x,t),\ t > 0, \quad u(x,0) = f(x)\ u_t(x,0) = g(x)
$$

we'd solve two PDEs. The first

$$
w\,u_{tt}^{(1)} + \mathscr{L}[u^{(1)}] = w(x)\,F(x,t),\ t > 0, \quad u^{(1)}(x,0) = u_t^{(1)}(x,0) = 0
$$

we'd solve with Duhamel's principle and the second

$$
w\,u_{tt}^{(2)} + \mathscr{L}[u^{(2)}] = 0,\ t > 0, \quad u^{(2)}(x,0) = f(x),\ u_t^{(2)}(x,0) = g(x)
$$

we'd solve with techniques from earlier in the course. Then $u(x,t) = u^{(1)}(x,t) + u^{(2)}(x,t)$ would solve the inhomogeneous PDE with inhomogeneous ICs.