Duhamel's Principle Example

Example 1. *Solve the inhomogeneous wave equation*

$$u_{tt} - c^2\, u_{xx} = F(x,t), \quad -\infty < x < \infty, \quad t > 0, \qquad u(x,0) = u_t(x,0) = 0$$

First note $w = 1$ in this example. Then, under Duhamel's principle we first solve the homogeneous PDE for $v(x,t;\tau)$

$$v_{tt} - c^2\, v_{xx} = 0, \quad -\infty < x < \infty, \quad t > \tau, \qquad v(x,\tau;\tau) = 0,\; v_t(x,\tau;\tau) = F(x,\tau).$$

The answer to this IVP is available to us via d'Alembert's solution.
d'Alembert's solution required the time domain $t > 0$ not $t > \tau$, we can fix this easily. Let $T = t - \tau$ then note that $v_t = v_T$ and $v_{tt} = v_{TT}$ and the IVP becomes

$$v_{TT}(x,T;\tau) - c^2\, v_{xx}(x,T;\tau) = 0, \quad -\infty < x < \infty, \quad T > 0, \qquad v(x,0;\tau) = 0,\; v_T(x,0;\tau) = F(x,\tau)$$

and d'Alembert's solution is then

$$v(x,T;\tau) = \frac{1}{2\,c} \int_{x-c\,T}^{x+c\,T} F(s,\tau)\, \mathrm{d}s$$

(the text in blue are steps that you do not need to explicitly show every time, you can account for this time shift "in your head", if you'd like)

$$v(x,t;\tau) = \frac{1}{2\,c} \int_{x-c\,(t-\tau)}^{x+c\,(t-\tau)} F(s,\tau)\, \mathrm{d}s$$

hence

$$u(x,t) = \frac{1}{2\,c} \int_0^t \int_{x-c\,(t-\tau)}^{x+c\,(t-\tau)} F(s,\tau)\, \mathrm{d}s\, \mathrm{d}\tau$$

solves the inhomogeneous wave equation in this example.
To extend the example, if we wanted the most general form with inhomogeneous ICs as well such as $u(x,0) = u_0(x)$ and $u_t(x,0) = v_0(x)$ then (following the process from last time) we'd superimpose two functions $u^{(2)}$ which solves the homogeneous PDE with inhomogeneous ICs and $u^{(1)}$ which solves the inhomogeneous PDE with homogeneous ICs. That is:

$$u(x,t) = \underbrace{\frac{1}{2}\left(u_0(x+c\,t) + u_0(x-c\,t)\right) + \frac{1}{2\,c}\int_{x-c\,t}^{x+c\,t} v_0(s)\, \mathrm{d}s}_{u^{(2)}} + \underbrace{\frac{1}{2\,c}\int_0^t \int_{x-c\,(t-\tau)}^{x+c\,(t-\tau)} F(s,\tau)\, \mathrm{d}s\, \mathrm{d}\tau}_{u^{(1)}}.$$

Summary

1. Duhamel's principle let's us go from solutions of a homogeneous IVP to solutions of an inhomogeneous problem

2. Intuitively, the inhomogeneous term (or source/forcing term) is something external being added to the system at each time step. Duhamel's principle solves (infinitely many) homogeneous IVPs where that source/forcing is instead thought of as an initial condition for a particular time step $t = \tau$. By adding all these up (integrating) we arrive at a final solution.

3. We saw Duhamel's principle for linear PDEs but it can also be applied to ODEs (and in doing is called "variation of parameters") and non-linear PDEs (for which the non-linearity is thought of as an inhomogeneity).

Inhomogeneous Boundary Conditions

So far all of our approaches have considered homogeneous boundary conditions. If instead we had a boundary conditions ike

$$\alpha(\vec{x})\,u(\vec{x},t)+\beta(\vec{x})\frac{\partial u(\vec{x},t)}{\partial n}\Big|_{\partial V}=B(\vec{x},t),\quad \vec{x}\in\partial V$$

then we decompose the solution into two parts

$$u(\vec{x},t)=W(\vec{x},t)+V(\vec{x},t)$$

where V is a function whose spatial dependency is simple (often linear in space) that satisfies the BCs and W is a function that solves the PDE with homogeneous BCs but with additional forcing terms. (In Duhamel's principle we traded forcing terms for inhomogeneous ICs, here we're trading inhomogeneous BCs for forcing terms).
To demonstrate this method, consider the heat equation

$$u_t-D\,u_{xx}=F(x,t),\quad 0<x<l,\quad t>0$$

with IBCs

$$u(x,0)=f(x),\quad 0<x<l,\quad u(0,t)=B_1(t),\quad u(l,t)=B_2(t),\quad t>0$$

let's focus on V. We want V to take care of the inhomogeneous BCs. So we'll define V to be the linear interpolant (in space) of our two BCs. Which is to say, V is the unique linear function such that $V(l,t)=B_2(t)$ and $V(0,t)=B_1(t)$. i.e.

$$V(x,t)=\frac{1}{l}\,(x\,B_2(t)+(l-x)\,B_1(t))$$

with V out of the way we have to figure out what the inhomogeneous PDE for W is. In general V won't solve the PDE (as is true in this case). So when we plug V into the LHS of the PDE we get some extra terms left over: these are the extra inhomogeneities. We now want to find the PDE that W sastisfies. Take $u(x,t)=W(x,t)+V(x,t)$ and sub it into the original PDE to get

$$\begin{aligned}u_t-D\,u_{xx}&=F(x,t)\\W_t+V_t-D\,W_{xx}-D\,V_{xx}&=F(x,t)\\W_t-D\,W_{xx}&=F(x,t)-V_t\\&=F(x,t)-\frac{1}{l}\,\big(x\,B_2'(t)+(l-x)\,B_1'(t)\big)\end{aligned}$$

while the RHS looks complicated, all it is just a different forcing term. To find the IC for W we take

$$\begin{aligned}u(x,0)&=f(x)\\W(x,0)+V(x,0)&=f(x)\\W(x,0)&=f(x)-\frac{1}{l}\,(x\,B_2(0)+(l-x)\,B_1(0))\end{aligned}$$

and, since V takes care of the BCs, the BCs for W are simply

$$W(0,t)=W(l,t)=0.$$

This equation in W is then an inhomogeneous PDE with homogeneous BCs – something we can solve with Duhamel's principle.
In general, if we want V to satisfy the BCs

$$\alpha_1\,V(0,t)-\beta_1\,V_x(0,t)=B_1(t),\quad \alpha_2\,V(l,t)+\beta_2\,V_x(l,t)=B_2(t)\qquad(1)$$

then take

$$V(x) = c_1\, x + c_2$$

and plug it into 1 which will give you a linear system of equations with two unknowns which you can solve for c_1 and c_2. Your system will have

$$\alpha_1\, \alpha_2\, l + \alpha_1\, \beta_2 + \alpha_2\, \beta_1$$

as it's determinant. So as long as this is nonzero, we can find a linear V that satisfies the BCs. If we require (as usual) that $l \neq 0$ and $(\alpha_i, \beta_i) \neq (0, 0)$ (as otherwise our domain is a point/we lose a BC), then the determinant is zero iff $\alpha_1 = \alpha_2 = 0$. In that case our BCs are Neumann

$$V_x(0, t) = \frac{B_1(t)}{-\beta_1}, \quad V_x(l, t) = \frac{B_2(t)}{\beta_2}$$

in which case we take V_x as the linear interpolant and integrate to arrive at V. In this case, our V is quadratic in space – but that's okay. It just adds different non-linearities to the RHS of the W PDE.