For a parabolic (or hyperbolic) PDE

$$w\,\frac{\partial u}{\partial t} + \mathscr{L}[u] = w\,F, \quad \alpha(\vec{x})\,u + \beta(\vec{x})\,\frac{\partial u}{\partial n}\Big|_{\partial V} = B(\vec{x},t),\, x\in\partial V, \quad u(\vec{x},0) = f(\vec{x}),\, \vec{x}\in V$$

first take $u = V + W$ and find V as the simple function that satisfies the BCs after which you'll have an inhomogeneous equation for W with homogeneous BCs. Then, find the solution for W by using Duhamel's principle to deduce a *new* homogeneous PDE for v. All this to say the "work" is in finding the solutions to the homogeneous PDE – everything after that is tricks to stitch together a solution to the inhomogeneous problem.

Method of Eigenfunction Expansion on Inhomogeneous PDEs

In separation of variables we solved homogeneous PDEs by finding solutions of the form

$$u(x,t) = \sum_{k=1}^{\infty} \hat{X}_k(x)\,T_k(t)$$

or

$$u(x,y) = \sum_{k=1}^{\infty} \hat{X}_k(x)\,Y_k(y)$$

our answer then involved the T_k or Y_k functions depending (via ICs/BCs) on projections of our initial data onto the eigenfunctions. Our solution for u was then called an eigenfunction expansion. Here we'll try and use this method of eigenfunction expansion on inhomogeneous PDEs. Before we see this method, note that

$$\begin{aligned}
u(x,y) &= \sum_{k=1}^{\infty} \hat{X}_k(x)\,Y_k(y)\\
w(x)\,\hat{X}_j(x)\,u(x,y) &= \sum_{k=1}^{\infty} w(x)\,\hat{X}_j(x)\,\hat{X}_k(x)\,Y_k(y)\\
\int_0^l w(x)\,\hat{X}_j(x)\,u(x,y)\,\mathrm{d}x &= \sum_{k=1}^{\infty} \int_0^l w(x)\,\hat{X}_j(x)\,\hat{X}_k(x)\,Y_k(y)\,\mathrm{d}x\\
\left(u, \hat{X}_j\right) &= \sum_{k=1}^{\infty} (\hat{X}_k, \hat{X}_j)\,Y_k(y)\\
\left(u, \hat{X}_j\right) &= Y_j(y)
\end{aligned}$$

As an example, we'll start with the elliptic PDE

$$-w(x)\,\frac{\partial^2 u(x,y)}{\partial y^2} + \mathscr{L}[u](x,y) = w(x)\,F(x,y)$$

For the homogeneous problems (in all three cases) we recovered the eigenvalue problem of the form

$$\mathscr{L}[\hat{X}_k](x) = w(x)\,\lambda_k\,\hat{X}_k(x)$$

Let $\hat{X}_k$ and λ_k be the eigenfunctions/eigenvalues for $(1/w)\mathscr{L}$. Then we multiply the inhomogeneous

PDE by $\hat{X}_k$ and integrate

$$-w\,\frac{\partial^2 u}{\partial y^2}\,\hat{X}_k + \mathscr{L}[u]\,\hat{X}_k = w\,F\,\hat{X}_k$$
$$-\int_0^l w\,\frac{\partial^2 u}{\partial y^2}\,\hat{X}_k\,\mathrm{d}x + \int_0^l \mathscr{L}[u]\,\hat{X}_k\,\mathrm{d}x = \int_0^l w\,F\,\hat{X}_k\,\mathrm{d}x$$
$$-\left(\frac{\partial^2 u}{\partial y^2}, \hat{X}_k\right) + \left(\frac{1}{w}\mathscr{L}[u], \hat{X}_k\right) = (F, \hat{X}_k)$$
$$\frac{\mathrm{d}^2}{\mathrm{d}y^2}\left(u, \hat{X}_k\right) = \underbrace{\left(\frac{1}{w}\mathscr{L}[u], \hat{X}_k\right) - (F, \hat{X}_k)}_{\text{functions of } y}$$

so this is really an ODE in terms of y – to make this more explicit we first note

$$\frac{\mathrm{d}^2 Y_k}{\mathrm{d}y^2} = \left(\frac{1}{w}\mathscr{L}[u], \hat{X}_k\right) - (F, \hat{X}_k)$$

we can simplify this further. In Lec 14 we saw that for homogeneous boundary conditions our operator $\frac{1}{w}\mathscr{L}$ was self-adjoint, meaning (with $v = \hat{X}_k$)

$$\left(\frac{1}{w}\mathscr{L}[u], \hat{X}_k\right) = \left(u, \frac{1}{w}\mathscr{L}[\hat{X}_k]\right)$$

hence, using the eigenvalue relation, we have

$$\left(\frac{1}{w}\mathscr{L}[u], \hat{X}_k\right) = \left(u, \frac{1}{w}\mathscr{L}[\hat{X}_k]\right) = \left(u, \lambda_k\,\hat{X}_k\right) = \lambda_k\,\left(u, \hat{X}_k\right) = \lambda_k\,Y_k$$

and so the y ODE becomes

$$\frac{\mathrm{d}^2 Y_k(y)}{\mathrm{d}y^2} = \lambda_k\,Y_k(y) - F_k(y)$$

where we've defined

$$F_k(y) = (F(x, y), \hat{X}_k(x))$$

for convenience.
This Y_k ODE also requires BCs which can be found via the boundary conditions for the original problem. i.e. if originally we had homogeneous Dirichlet BCs on a square domain, then $u(x, 0) = u(x, l) = 0 \implies Y_k(0) = Y_k(l) = 0$.
In summary: We can find solutions of inhomogeneous, elliptic PDEs by first finding the eigenfunctions $\hat{X}_k$ of $(1/w)\mathscr{L}$ and then projecting our PDE onto these eigenfunctions to deduce an ODE for Y_k. This procedure also works for parabolic and hyperbolic PDEs, in which case we end up with the ODEs

$$\frac{\mathrm{d}T_k(t)}{\mathrm{d}t} = -\lambda_k\,T_k(t) + F_k(t), \quad T_k(0) = (f(x), \hat{X}_k(x))$$

and

$$\frac{\mathrm{d}^2 T_k(t)}{\mathrm{d}t^2} = -\lambda_k\,T_k(t) + F_k(t), \quad T_k(0) = (f(x), \hat{X}_k), \quad T_k'(0) = (g(x), \hat{X}_k)$$

Summary

- In eigenfunction expansion for forced PDEs we expand both the source (F) and the solution (u) in terms of the eigenfunctions from the homogeneous problem. This gives us ODEs for the coefficient functions in the expansion of u
- Duhamel's principle is applied to initial value problems, the eigenfunction expansion method can be used for any and all problems (including elliptic types).