Eigenfunction Example: Resonance

Consider solving the following inhomogeneous hyperbolic equation via the method of eigenfunction expansion:

$$w(x)\, u_{tt}(x,t) + \mathscr{L}[u](x,t) = w(x)\, \hat{X}_j(x)\, \sin(\omega\, t), \quad u(x,0) = u_t(x,0) = 0$$

where $\hat{X}_j$ is an eigenfunction of $(1/w)\,\mathscr{L}$. That is, the PDE is being forced with the spatial structure of an eigenfunction (and temporal part a sinusoid with radian frequency ω).
In the notation from last time,

$$F_k(t) = (F(x,t), \hat{X}_k) = (\hat{X}_j(x)\, \sin(\omega\, t), \hat{X}_k(x)) = \sin(\omega\, t)\, (\hat{X}_j(x), \hat{X}_k(x)) = \sin(\omega\, t)\, \delta_{j,k}.$$

We saw that the method of eigenfunction expansions reduced to solving an ODE of the form

$$\frac{\mathrm{d}^2 T_k}{\mathrm{d}t^2} + \lambda_k\, T_k = F_k, \quad T_k(0) = (f(x), \hat{X}_k), \quad T_k'(0) = (g(x), \hat{X}_k) \quad k = 1, 2, \ldots$$

where λ_k is the eigenvalue to $(1/w)\mathscr{L}$ associated with $\hat{X}_k$.
In this example $f(x) = g(x) = 0$ so the ICs to the ODE are $T_k(0) = T_k'(0) = 0$. Thus, the only thing keeping the ODE from having non-trivial solutions is the forcing term/inhomogeneity $F_k(t)$. But we showed that $F_k(t) = 0$ for all $k \neq j$ and $F_j(t) = \sin(\omega\, t)$. Thus $T_k(t) = 0$ for all $k \neq j$. We only need to solve one ODE to find the solution to our inhomogeneous PDE!
We focus on

$$\frac{\mathrm{d}^2 T_j}{\mathrm{d}t^2} + \lambda_j\, T_j = F_j = \sin(\omega\, t), \quad T_j(0) = T_j'(0) = 0. \tag{1}$$

This ODE is inhomogeneous. We solve it via Laplace Transforms.

Brief Laplace Transform Review

Recall (from AMATH 250/251 or similar) that for the Laplace transform $\mathcal{L}$ we have (for constants c_1, c_2, and ω and arbitrary functions F and G)

$$\begin{aligned}
\mathcal{L}[c_1\, F(t) + c_2\, G(t)] &= c_1\, \mathcal{L}[F] + c_2\, \mathcal{L}[G] && \text{Sim. for } \mathcal{L}^{-1} \\
\mathcal{L}[y'] &= s\, \mathcal{L}[y] - y(0) \\
\mathcal{L}[y''] &= s^2\, \mathcal{L}[y] - s\, y(0) - y'(0) \\
\mathcal{L}^{-1}\left[\mathcal{L}[F]\right] &= F \\
\mathcal{L}^{-1}\left[\mathcal{L}[F]\, \mathcal{L}[G]\right] &= F * G = \int_0^t F(t-\tau)\, G(\tau)\, \mathrm{d}\tau \\
\mathcal{L}[\sin(\omega\, t)] &= \frac{\omega}{\omega^2 + s^2}
\end{aligned}$$

Hence to solve our inhomogeneous ODE (1) we proceed by taking the Laplace transform of both sides to see:

$$\begin{aligned}
\mathcal{L}\left[\frac{\mathrm{d}^2 T_j}{\mathrm{d}t^2}\right] + \lambda_j\, \mathcal{L}\left[T_j\right] &= \mathcal{L}\left[F_j\right] \\
s^2 \mathcal{L}\left[T_j\right] - s\, T_j(0) - T_j'(0) + \lambda_j\, \mathcal{L}\left[T_j\right] &= \mathcal{L}\left[F_j\right] \\
(s^2 + \lambda_j)\, \mathcal{L}\left[T_j\right] &= \mathcal{L}\left[\sin(\omega\, t)\right] \\
\mathcal{L}[T_j] &= \frac{1}{s^2 + \lambda_j}\, \frac{\omega}{\omega^2 + s^2}
\end{aligned}$$

hence, by inverting the Laplace transform of both sides, we have

$$\begin{aligned} T_j(t) &= \mathcal{L}^{-1}\left[\frac{1}{\sqrt{\lambda_j}}\frac{\sqrt{\lambda_j}}{s^2+\lambda_j}\right] * \mathcal{L}^{-1}\left[\frac{\omega}{\omega^2+s^2}\right] \\ &= \frac{1}{\sqrt{\lambda_j}}\mathcal{L}^{-1}\left[\frac{\sqrt{\lambda_j}}{s^2+\lambda_j}\right] * \mathcal{L}^{-1}\left[\frac{\omega}{\omega^2+s^2}\right] \\ &= \frac{1}{\sqrt{\lambda_j}} \sin\left(\sqrt{\lambda_j}\, t\right) * \sin(\omega\, t) \\ &= \frac{1}{\sqrt{\lambda_j}} \int_0^t \sin(\sqrt{\lambda_j}\,(t-\tau))\, \sin(\omega\,\tau)\,\mathrm{d}\tau \\ &= \frac{1}{\sqrt{\lambda_j}} \left(\frac{\omega\, \sin(\sqrt{\lambda_j}\, t) - \sqrt{\lambda_j}\, \sin(\omega\, t)}{\omega^2-\lambda_j}\right). \end{aligned}$$

Putting it all together our final solution is

$$u(x,t) = \sum_{k=1}^{\infty} \hat{X}_k(x)\, T_k(t) = \hat{X}_j(x)\, T_j(t) = \frac{1}{\sqrt{\lambda_j}}\left(\frac{\omega\, \sin(\sqrt{\lambda_j}\, t) - \sqrt{\lambda_j}\, \sin(\omega\, t)}{\omega^2-\lambda_j}\right)\, \hat{X}_j(x).$$

Now ω was unspecified – so we would hope for a solution for all $\omega \in \mathbb{R}$. The problem is: what about $\omega = \sqrt{\lambda_j}$? In that situation we take

$$\begin{aligned} u(x,t) &= \lim_{\omega\to\sqrt{\lambda_j}} \frac{1}{\sqrt{\lambda_j}}\left(\frac{\omega\, \sin(\sqrt{\lambda_j}\, t) - \sqrt{\lambda_j}\, \sin(\omega\, t)}{\omega^2-\lambda_j}\right)\, \hat{X}_j(x) \\ &= \frac{1}{2\,\sqrt{\lambda_j}}\left(\frac{\sin(\sqrt{\lambda_j}\, t)}{\sqrt{\lambda_j}} - t\, \cos(\sqrt{\lambda_j}\, t)\right)\, \hat{X}_j(x) \end{aligned}$$

with a little bit of help from l'Hôpital.

What does this mean? In the first case ($\omega \neq \sqrt{\lambda_j}$), we have that forcing via an eigenfunction collapses our solution down to a projection onto a single eigenfunction. i.e. the solution's spatial structure is dictated by that forcing eigenfunction. If additionally, we have that our forced solution oscillates in time at the frequency associated with the forcing eigenfunction ($\omega = \sqrt{\lambda_j}$), then not only is the spatial structure completely dicated by the eigenfunction but the solution grows quasi-linearly in time! (In the sense that solutions are still oscillatory in time, but the amplitude of those oscillations is growing linearly).