Quasi-Linear PDEs

So far in the course we've considered only linear PDEs. The class of all non-linear PDEs is very large and beyond the scope of this course (see: AMATH 453), however we'll consider one particular type of non-linear PDE:

Definition 1 (Quasi-Linear PDE)**.** *A quasi-linear PDE is like a linear PDE except the coefficient functions can additionally depend upon u and lower order derivatives of u.*

i.e. for first order PDEs we're looking at PDEs of the form

$$a(x,t,u)\,u_t + b(x,t,u)\,u_x = c(x,t,u)$$

for second order PDEs we're considering the form

$$a(x,t,u,u_t,u_x)\,u_{tt} + 2\,b(x,t,u,u_t,u_x)\,u_{tx} + c(t,x,u,u_t,u_x)\,u_{xx} = d(x,t,u,u_t,u_x)$$

As an illustrative example consider the non-linear wave equation

$$u_{tt} - \frac{c_0^2}{(1+u_x^2)^{3/2}}\,u_{xx} = 0.$$

(We derived the linear wave equation in Lec 5 by assuming $u_x^2 \approx 0$, if instead we left $u_x^2 \neq 0$, we'd have arrived at the non-linear form above) This is an example of a quasi-linear PDE. Really, this is just the wave equation with

$$c(u_x) = \frac{c_0}{(1+u_x^2)^{3/4}}.$$

Intuitively this definition is saying: the flatter a wave, the greater it's speed (up to a maximum of $c = c_0$ when $u_x = 0$). The steeper a wave, the slower it is (to a minimum of $c = 0$ when $u_x \to \pm\infty$).

If you've ever surfed, this should match your intuition: as waves begin to form they're fast, then as the wave gets larger (and steeper), the mass of water it has to move becomes greater and the wave slows down. Eventually the wave gets so steep that the slope becomes vertical, at which point the wave "breaks" and starts folding forward (and this is what allows you to surf). Mathematically, a wave breaking occurs when $u_x \to \infty$. In the example of a plucked string, this occurs when the displacements in the string get so large that the string becomes vertical. The occurrence of a vertical string is called a *shock*. Shocks (or shock waves) and shock formation is an important aspect of non-linear PDE analysis and is what we'll focus on this week.

Solution's via Method of Characteristics

Consider the (easier) 1st order inviscid Burgers' equation

$$u_t + u\,u_x = 0, \quad -\infty < x < \infty, \quad t > 0, \quad u(x,0) = f(x)$$

(Note that this is just the linear advection equation with $c = u$ – remember that the linear advection equation was "one half" of the wave equation). For now, we'll assume that f is continuous with continuous first derivative.

In the (conventional) method of characteristics approach, we would assume that there are curves $x(t)$ in space-time such that u is constant along these curves. This assumption gives us

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\left(u(x(t),t)\right) = u_x\,x'(t) + u_t$$

hence,

$$x'(t) = u$$

is the characteristic ODE.
Now *along characteristic curves* the solution u is a constant. The characteristic ODE has an implicit solution $\psi(x,t) = k$. For each k, a single characteristic curve is defined. Along that *particular* characteristic curve $u(x,t) = u_k$ is a constant. (i.e. for $k = 1$ you'll get *one* characteristic curve along which u takes on some particular value u_1, for $k = 2$ you'll get a *different* characteristic curve along which u takes on a *different* constant value u_2). The characteristic curves are defined by this characteristic ODE. Therefore, in the characteristic ODE, u is a constant (which I'll call u_k). More formally if $\psi(x,t) = k$ is the solution of the characteristic ODE then:

$$u(x,t) = u_k \in \mathbb{R} \quad \text{if } (x,t) \text{ are chosen such that } \psi(x,t) = k.$$

However the characteristic ODE *defines* such a ψ. That is, x and t are along the characteristic. Hence the characteristic ODE is really

$$x'(t) = u_k \implies x(t) = u_k\, t + k$$

or,

$$\psi(x,t) = x - u_k\, t$$

suggesting the change of variables

$$\xi = x - u_k\, t, \quad \eta = t.$$

Briefly, let's check the Jacobian determinant of this transformation (along the characteristic)

$$|J(\xi,\eta)| = \begin{vmatrix} \xi_x & \xi_t \\ \eta_x & \eta_t \end{vmatrix} = \begin{vmatrix} 1 & -u_k \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

We proceed like normal: we use u to refer to the solution in physical coordinates $u(x,t)$ and $\hat{u}$ to refer to the solution in characteristic coordinates $\hat{u}(\xi,\eta)$, however $\hat{u}(\xi,\eta) = u(x,t)$. u_k is just a constant so we don't bother making a distinction. We calculate

$$\begin{aligned} u_t &= \hat{u}_\xi\, \xi_t + \hat{u}_\eta\, \eta_t \; = -u_k\, \hat{u}_\xi + \hat{u}_\eta \\ u_x &= \hat{u}_\xi\, \xi_x + \hat{u}_\eta\, \eta_x = \hat{u}_\xi \end{aligned}$$

which we substitute into our PDE to see

$$\begin{aligned} u_t + u\, u_x &= (-u_k\, \hat{u}_\xi + \hat{u}_\eta) + \hat{u}\,(\hat{u}_\xi) \\ &= \hat{u}_\eta + (\hat{u} - u_k)\, \hat{u}_\xi \\ &= 0. \end{aligned}$$

Again if we assume x and t are along our characteristic $\psi = k$, then the PDE simplifies to

$$\hat{u}_\eta = 0$$

which we can solve.