Solution of Inviscid Burgers' Equation

We reduced the inviscid Burgers' equation

$$u_t + u\, u_x = 0$$

via the change of variables $\xi = x - u_k\, t$, $\eta = t$ to

$$\hat{u}_\eta = 0$$

which can be immediately solved as

$$\hat{u}(\xi, \eta) = F(\xi)$$

or

$$u(x,t) = F(x - u_k\, t) = F(x - u(x,t)\, t).$$

If we try to incorporate the IC we have

$$u(x,0) = F(x - u(x,0)\, 0) = F(x) = f(x)$$

hence (in *this* situation) we have $F(x) = f(x)$. So our final (implicit) solution to the inviscid Burgers' equation is

$$u(x,t) = f(x - u(x,t)\, t).$$

We can verify that this solution is accurate by first recognizing that it satisfies the IC and then by taking the time derivative of both sides to see

$$\begin{aligned} \frac{\partial}{\partial t}\left(u(x,t)\right) &= \frac{\partial}{\partial t}\left(f(x - u(x,t)\, t)\right) \\ u_t &= f'(x - u\, t)\, \left(-u_t\, t - u\right) \\ u_t &= -f'(x - u\, t)\, t\, u_t - f'(x - u\, t)\, u \\ u_t &= \frac{-f'(x - u\, t)\, u}{1 + t\, f'(x - u\, t)} \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{\partial}{\partial x}\left(u(x,t)\right) &= \frac{\partial}{\partial x}\left(f(x - u(x,t)\, t)\right) \\ u_x &= f'(x - u\, t)\, \left(1 - u_x\, t\right) \\ u_x &= \frac{f'(x - u\, t)}{1 + t\, f'(x - u\, t)}. \end{aligned}$$

Hence

$$u_t + u\, u_x = \frac{-f'(x - u\, t)\, u}{1 + t\, f'(x - u\, t)} + u\, \frac{f'(x - u\, t)}{1 + t\, f'(x - u\, t)} = 0$$

and so the PDE is solved.

Our solution is called the *implicit* solution for u – we won't always be able to get an *explicit* solution. Though for some initial conditions we may! Consider the initial profile $f(x) = a + b\, x$. In this case, that would mean

$$u(x,t) = f(x - u\, t) = a + b\, x - b\, u\, t \implies u(x,t) = \frac{b\, x + a}{b\, t + 1}.$$

In which case we have a single, explicit definition of u. Now explicit solutions are more desirable than implicit ones, sure, but implicit solutions contain everything we'd ever need. Most ODE solution techniques you saw in previous courses resulted in implicit solutions. Moreover, in general if you have an implicit solution

$$u(x,t) = F(x,t,u(x,t))$$

you can numerically construct the *entire* solution for $t \geq 0$, starting from our initial point, via a straight-forward approach (based on Euler's method).

0. Choose a time step Δt
1. Take $u(x,0) = f(x)$ as given
2. Take $u(x,\Delta t) = F(x,\Delta t, u(x,0))$ (which can be calculated because we know x, Δt, and $u(x,0)$).
3. Given $u(x, n\,\Delta t)$ to find $u(x,(n+1)\,\Delta t)$, take $u(x,(n+1)\,\Delta t) = F(x,(n+1)\,\Delta t, u(x,n\,\Delta t))$. Which *can* be calculated because we know everything on the RHS.

Even for "simple" initial conditions, like $f(x) = 1 - x$, the solution is

$$u(x,t) = \frac{1-x}{1-t}$$

which has a clear singularity at $t = 1$.
In the linear PDE case, we argued that the method of characteristics provided a unique solution by arguing two points: our characteristic ODE was always solvable and our change of variables was always invertible. If we make the same assumptions about smoothness of coefficient functions $a(x,t,u)$ and $b(x,t,u)$, then we have that the characteristic ODE is solvable. So let's re-examine the Jacobian step. Along the characteristic we saw that $|J(\xi,\eta)| = 1$. Suppose we weren't restricted to keeping x and t along the same characteristic, and instead let them vary freely. Now if I fix t and vary x, then I'm always intersecting *a* characteristic, it just might not be the same one for every x. That is, the k is changing (and then, so too is the constant value of u). As a reminder: $\eta = t$ and $\xi = x - u\,t$. Then

$$|J(\xi,\eta)| = \begin{vmatrix} \xi_x & \xi_t \\ \eta_x & \eta_t \end{vmatrix} = \begin{vmatrix} 1 - u_x\,t & -u - u_t \\ 0 & 1 \end{vmatrix} = 1 - u_x\,t$$

For our initial time $t = 0$ we have that the Jacobian is non-zero. Hence, by implicit function theorem, there is *neighbourhood* about $t = 0$ for which the Jacobian is non-zero and finite. That is, we're not guaranteed that the change of variables is invertible for all time, just for a neighbourhood about $t = 0$. Thus we're not guaranteed that this method will always work: there are places where our solution will break down, beyond which we are not guaranteed that our solution is trustworthy. This is due to the presence of shocks.
For the $f(x) = 1 - x$ IC we had $u_x = \frac{-1}{1-t}$ at which point the Jacobian determinant is

$$|J| = 1 + \frac{1}{1-t}\,t = \frac{1}{1-t}$$

which is not defined at $t = 1$. Hence, at $t = 1$ our change of variables is not guaranteed to be invertible. Let's plot some characteristic curves for this IC. Our characteristics are defined by $\psi(x,t) = k$ for the inviscid Burgers' equation this is

$$x - u\,t = k$$

Hence in x-t space these are functions

$$t = \frac{1}{u_k}(x - k).$$

Thus our characteristics are straight lines with slope $1/u_k$. Now this feels like we have too many variables to actually solve for this without using the implicit solution. While we have an explicit solution for this IC, we won't always. But! We just need to focus on the IC. At $(x,t) = (x_k, 0)$ we have $u_k = u(x,t) = u(x_k,0) = f(x_k) = 1 - x_k$. Moreover, we can determine x_k by evaluating our characteristics at $(x,t) = (x_k,0)$ to see

$$0 = \frac{1}{u_k}(x_k - k) \implies x_k = k$$

Hence u_k is known to us and so the IC lets us completely determine the shapes of the characteristics. That is, for any IC, the characteristics of the inviscid Burgers' equation are

$$t = \frac{1}{f(k)}(x - k)$$

for our particular IC this is

$$t = \frac{x-k}{1-k}. \tag{1}$$

(As an aside, if you plug in this value of t into $u(x,t)$ you should get $u_k = 1 - k$: demonstrating that solutions are constant along characteristics!) Contrast this with the linear PDE case: in linear PDEs, whenever we had straight-line characteristics, they were always parallel for different k. Here, the slope depends upon k and so that is not true. For first order, linear PDEs our characteristics never intersected (because our Jacobian was always a finite, non-zero number). In the non-linear case, that is not always true.