Shocks in the Inviscid Burgers' Equation

We saw that characteristics obey

$$t = \frac{x-k}{1-k} \tag{1}$$

and argued, graphically, that all characteristics intersect.
For instance, suppose $k_1 \neq k_2$

$$\frac{x-k_1}{1-k_1} = \frac{x-k_2}{1-k_2} \iff x = 1.$$

Given that k_1 and k_2 are arbitrary, all of the characteristics intersect at $t = 1$ (we've already seen that our Jacobian is infinite at $t = 1$ and that our solution is undefined at $t = 1$ for this IC). What does this mean for our solution u? Along characteristics u is supposed to be constant. At $t = 1$ what value does u take? There are infinitely many characteristics to choose from (and so infinitely many "values" of u). In this case $t = 1$ is an example of a **shock** and we say $t_s = 1$ (the time location of the shock/breaking time of the system). Similarly, we can find the place in space where the shock occurs. We do this by substituting t_s into (1) which, in this case, yields $x = 1$. Thus the spatial location of the shock is $x_s = 1$.
In general, for a 1D non-linear PDE a **shock** occurs when $u_x(x_s, t_s) \to \pm\infty$ – that is, when the solution becomes vertical. In the case of the inviscid Burgers' equation we can derive the values of x_s and t_s from this relation

$$u_x = \frac{f'(x-u\,t)}{1+t\,f'(x-u\,t)}$$

remember f is my initial data and is a C^1 function (by assumption). Since f is C^1, we have that f' is finite. Thus, the only way $u_x \to \pm\infty$ is if the denominator tends to zero. So if $f'(x) > 0$ for all $x \in \mathbb{R}$, then u_x is always finite and no shocks occur (for $t > 0$). (Easy enough to see that for $f(x) = 1 + x$ that the shock occurs at $(x_s, t_s) = (-1, -1)$). The slope $u_x \to \pm\infty$ as

$$t \to -\frac{1}{f'(k)}$$

hence if $f'(x_0) < 0$ for any x_0, then the solution along the characteristic issuing from the point $(x_0, 0)$ will break down at time $-1/f'(x_0)$. As a consequence, the *earliest* breaking time (or the temporal location of the first shock) is

$$t_s^* = \min_{k|f'(k)<0}\left(-\frac{1}{f'(k)}\right).$$

In english: the time of the first shock is found by minimizing $-1/f'(k)$ over all k values that keep this fraction positive. (Since we only care about positive time.)

A*nother* way to see the same thing is to find when the characteristics intersect. Let k be given and $k_2 = k + \epsilon$. Suppose these two characteristics intersect. Then,

$$k + f(k)\,t = k_2 + f(k_2)\,t$$

solving for t gives us

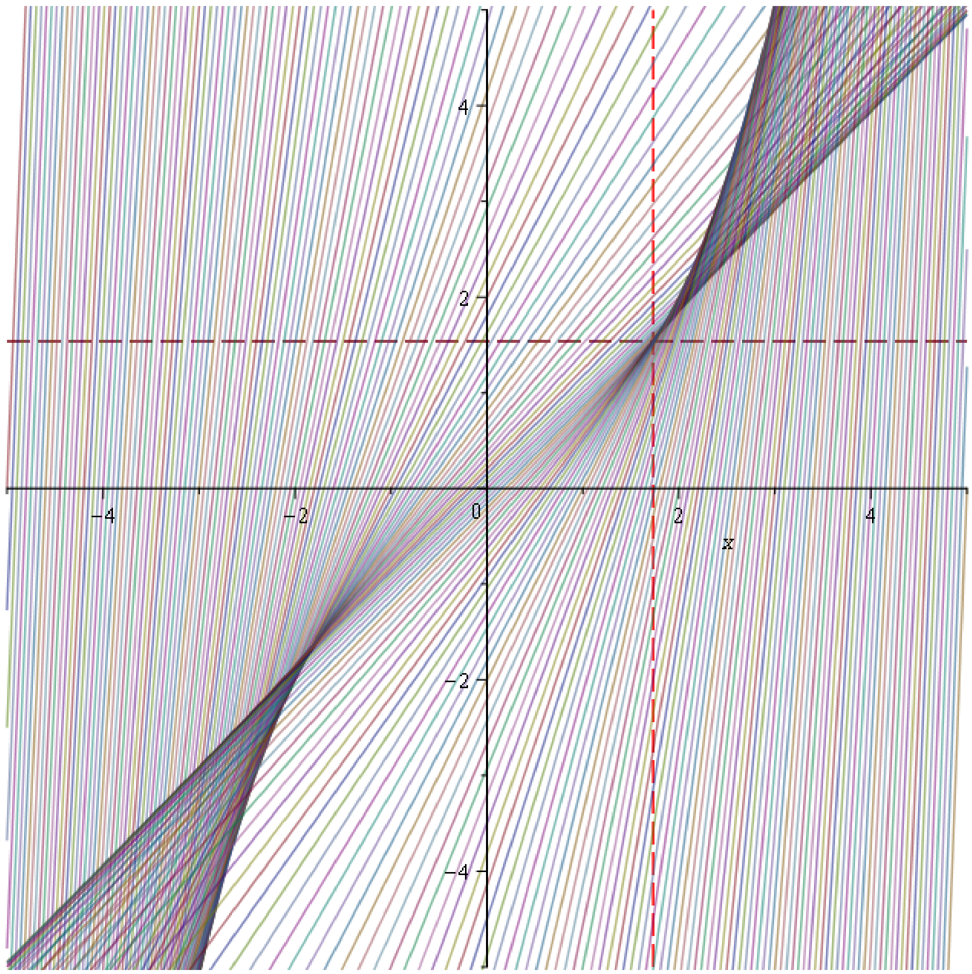
$$t = -\frac{k_2 - k}{f(k_2) - f(k)} = -\frac{\epsilon}{f(k+\epsilon) - f(k)}$$

we now want two things: we want k to represent the *first* characteristic that intersects another, so we should choose k such that t above is minimized (but positive, by assumption). Second: we want k_2 (the characteristic that intersects k) to be the *next* such characteristic. That is, we take $\epsilon \to 0$ in doing so we recover

$$t_s^* = \min_{\xi|f'(\xi)<0}\left(-\frac{1}{f'(\xi)}\right).$$

The first IC we looked at was a relatively simple IC. As a result, we only had one shock – one place where our characteristics intersected. In general, that is not the case. For instance, if we take the $f(x) = 1/(1+x^2)$ IC instead, then by (1) we have

$$t = (1+k^2)(x-k)$$



as the equation of our characteristics and in order to find the time of the first shock we need to minimize

$$-\frac{1}{f'(k)} = \frac{(k^2+1)^2}{2\,k}$$

over all the k values that keep the quantity positive (i.e. all $k > 0$ in this case). You can do this using any of your methods from a first year calculus sequence. For convenience define $g(k) = -1/f'(k)$ then, in this case, we want to minimize $g(k)$ on the domain $(0, \infty)$. Well,

$$g'(k) = \frac{3\,k^4 + 2\,k^2 - 1}{2\,k^2}$$

the numerator is a quadratic in k^2 with roots $k^2 = -1$ and $k^2 = 1/3$, hence $g'(k) = 0$ for $k = \pm i$ or $k = \pm\frac{1}{\sqrt{3}}$. Therefore $k_{\min} = 1/\sqrt{3}$ and

$$t_s = -\frac{1}{f'\left(\frac{1}{\sqrt{3}}\right)} = \frac{8}{9}\sqrt{3} \approx 1.54.$$

To find the x_s^* location, we substitute $k = k_{\min}$ in to the equation for our characteristic (intuitively, the $k_{\min}$ value tells us the characteristic that hits the shock first, along a characteristic as long as we know

t then we can find x, so we use our value of t_s to "trace" our characteristic to find x_s^*).

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\begin{aligned}
t_s &= \frac{1}{f(k_{\text{min}})}\,(x_s^* - k_{\text{min}}) \\
t_s &= (1 + k_{\text{min}}^2)\,(x_s^* - k_{\text{min}}) \\
\implies x_s^* &= \sqrt{3}
\end{aligned}
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