Shocks in Second (or higher) order Quasi-Linear PDEs

For higher order quasilinear PDEs proceed like we did with linear PDEs – first factor the operator, then solve via method of characteristics. This is the basic approach, but interesting things happen when shocks defined by completely different first order operators collide - analysis of these types of PDEs is put aside until 453.

Shocks Summary

Shocks are locations in space-time

- where our solution obtains a vertical slope
- where our mathematical model no longer represents physical reality
- beyond which the validity of our solution is suspect.

(On the domain $0 \le t < t_s^*$, $-\infty < x < \infty$ our solution is definitely trustworthy.)

Expansion Fans

Another interesting phenomenon, completely separate from shocks, is called an **expansion fan**. To illustrate this, consider the inviscid Burgers' equation with a step function as its initial profile

$$u_t + u\,u_x = 0, \quad u(x,0) = f(x) = \begin{cases} B & x \le a \\ A & x > a \end{cases},$$

where $B < A$ are known constants. We know that the implicit solution is

$$u(x,t) = f(x - u(x,t)\,t)$$

now f is a relatively simple function,

$$u(x,t) = \begin{cases} B & x - u\,t \le a \\ A & x - u\,t > a \end{cases}$$

which simplifies to

$$u(x,t) = \begin{cases} B & x \le a + t\,B \\ A & x > a + t\,A \end{cases}.$$

There's a problem here, to illustrate it more deeply consider $a = 0$, $B = 1$, $A = 2$. Then,

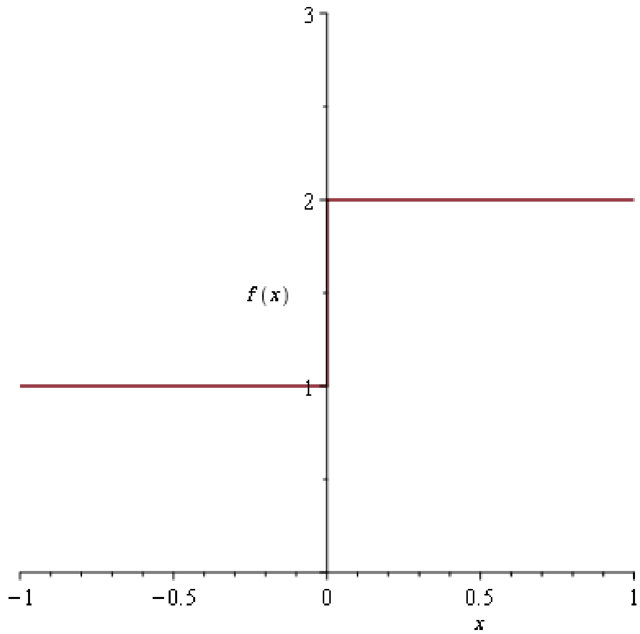
$$u(x,t) = \begin{cases} 1 & x \le t \\ 2 & x > 2\,t \end{cases}$$

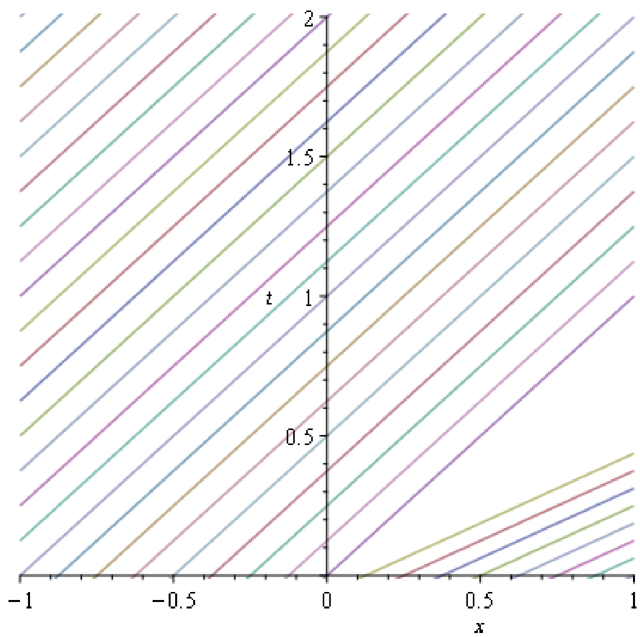
which raises the natural question: what value is $u(0.75, 0.5)$? It is undefined by our current solution. In fact, any value of x such that

$$a + t\,B < x \le a + t\,A$$

is undefined in our current solution (which is a very large swathe of space-time!).
From before we know the characteristics are given by

$$t = \frac{1}{f(k)}(x - k) = \begin{cases} \frac{1}{B}\,(x-k) & k \le a \\ \frac{1}{A}(x-k) & k > a \end{cases}$$





To fill in the void we insert what is known as an *expansion fan*. We define the fan as

$$\phi(x,t) = \frac{x - x_0}{t - t_0}$$

for constant values of x_0 and t_0. (Which we'll find next time)