Previously we saw that for a discontinuous initial condition, we had positions in space-time where there are *no* characteristics (and so no solution). To fill in the void we insert what is known as an *expansion fan*. We define the fan as

$$\phi(x,t) = \frac{x - x_0}{t - t_0}$$

for constant values of x_0 and t_0. (Which we'll find next time) which solves the PDE (easy to check). We must choose x_0 and t_0 to satisfy continuity conditions.

$$\begin{aligned} \phi(a + t\,A, t) &= A \quad \text{for all } t \\ \phi(a + t\,B, t) &= B \quad \text{for all } t \end{aligned}$$

i.e.

$$\begin{aligned} a + t\,A - x_0 &= t\,A - t_0\,A \\ a + t\,B - x_0 &= t\,B - t_0\,B \end{aligned}$$

subtracting the two gives $t_0(A - B) = 0 \implies t_0 = 0$ and putting this back into either of the above equations yields $x_0 = a$. Hence, our expansion fan is

$$\phi(x,t) = \frac{x - a}{t}$$

which fills in the gaps between the characteristics. This expansion fan is the continuous interpolation of our solution in this problem area. Expansion fans arise due to a discontinuity in our initial data. While this may seem like a simple enough idea, an extension (called the Prandtl–Meyer fan) is *very* useful in aerodynamics.

All told, then, our solution is

$$u(x,t) = \begin{cases} B & x \le a + t\,B \\ \frac{x-a}{t} & a + t\,B < x \le a + t\,A \\ A & x > a + t\,A \end{cases}.$$

Fourier Transform and Separation of Variables

For linear PDEs on finite spatial domains, the method of separation of variables is quite powerful. It's natural to try and adapt this method from finite domains to infinite ones. We will spend the rest of the course investigating something interesting that happens when we do so. We'll extend the method of separation of variables to infinite domains. Along the way we'll see we need some generalization of the Fourier series to integrals. This will result in the Fourier Transform. The Fourier Transform, it turns out, reduces solving infinite domain PDEs down to solving reduced, easier problems (either algebraic equations or ODEs). This transform is so useful, we don't tend to actually use the method of separation of variables on infinite domains anyway (as doing so is equivalent to solving via Fourier Transform).

The fundamental difference between finite domain problems and infinite domain problems is the number of eigenvalues. In finite domains, we had countably infinitely many eigenvalues. That is, the spectrum was discrete

$$0 \le \lambda_0 < \lambda_1 < \cdots$$

and we wrote our solution as

$$u(x,t) = \sum_{k=0}^{\infty} T_k(t)\,\hat{X}_k(x)$$

In contrast, on *infinite* domains, we have uncountably infinitely many eigenvalues. That is, the spectrum is continuous and we write our solution as

$$u(x,t) = \int_0^{\infty} T_\lambda(t)\,\hat{X}_\lambda(x)\,\mathrm{d}\lambda = \int_{-\infty}^{\infty} T_\omega(t)\,\hat{X}_\omega(x)\,\mathrm{d}\omega$$

If we perform separation of variables by assuming $u = X(x)\,T(t)$, we've seen that in all three cases we deal with an eigenvalue problem

$$\frac{1}{w(x)}\mathscr{L}[X](x) = \lambda\,X(x) = \omega^2\,X(x)$$

(where I'm using Sturm-Liouville theory that says that eigenvalues are non-negative).
Consider a really simple form of this problem where $w = 1$, $p = 1$, and $q = 0$ (as in the case of the prototypical equations), then $\mathscr{L} = -\nabla^2$ and

$$X'' + \omega^2\,X = 0, \qquad -\infty < x < \infty$$

Which we can immediately solve as

$$\hat{X}_\omega(x) = a(\omega)\,e^{i\,\omega\,x} + b(\omega)\,e^{-i\,\omega\,x}$$

for some $a(\omega)$ and $b(\omega)$ that normalize the function – (a different constant for each $\omega \in \mathbb{R}$, hence a and b are functions of ω). Now, at this step we can't completely determine the form of a and b because we do not have any boundary conditions on an infinite domain. Hence we'll need to use the IC at this step. Suppose we had to satisfy some IC $u(x,0) = f(x)$ then, in any of the three cases, we have

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} T_\omega(0)\,\hat{X}_\omega(x)\,\mathrm{d}\omega \\ &= \int_{-\infty}^{\infty} T_\omega(0)\,a(\omega)\,e^{i\,\omega\,x} + T_\omega(0)\,b(\omega)\,e^{-i\,\omega\,x}\,\mathrm{d}\omega \\ &= \int_{-\infty}^{\infty} \alpha(\omega)\,e^{i\,\omega\,x} + \beta(\omega)\,e^{-i\,\omega\,x}\,\mathrm{d}\omega \end{aligned}$$

Now this is also true under the transformation $\omega \to (-\tilde{\omega})$ which yields

$$\begin{aligned} f(x) &= -\int_{\infty}^{-\infty} \alpha(-\tilde{\omega})\,e^{-i\,\tilde{\omega}\,x} + \beta(-\tilde{\omega})\,e^{i\,\tilde{\omega}\,x}\,\mathrm{d}\tilde{\omega} \\ &= \int_{-\infty}^{\infty} \alpha(-\tilde{\omega})\,e^{-i\,\tilde{\omega}\,x} + \beta(-\tilde{\omega})\,e^{i\,\tilde{\omega}\,x}\,\mathrm{d}\tilde{\omega} \end{aligned}$$

adding these two together gives

$$2\,f(x) = \int_{-\infty}^{\infty} (\alpha(\omega) + \beta(-\omega))\,e^{i\,\omega\,x} + (\alpha(-\omega) + \beta(\omega))\,e^{-i\,\omega\,x}\,\mathrm{d}\omega$$

Let $\gamma(\omega) = \alpha(\omega) + \beta(-\omega)$, then we have

$$2\,f(x) = \int_{-\infty}^{\infty} \gamma(\omega)\,e^{i\,\omega\,x}\,\mathrm{d}\omega + \int_{-\infty}^{\infty} \gamma(-\omega)\,e^{-i\,\omega\,x}\,\mathrm{d}\omega = \int_{-\infty}^{\infty} \gamma(\omega)\,e^{i\,\omega\,x}\,\mathrm{d}\omega + \int_{-\infty}^{\infty} \gamma(\tilde{\omega})\,e^{i\,\tilde{\omega}\,x}\,\mathrm{d}\tilde{\omega}$$

therefore

$$f(x) = \int_{-\infty}^{\infty} \gamma(\omega)\,e^{i\,\omega\,x}\,\mathrm{d}\omega.$$

for some, as of yet unknown function, $\gamma(\omega)$. This is an example of an integral transform, an operation that changes a function of one variable (in this case, $\gamma(\omega)$) and translates it into a function of another (in this case, $f(x)$). These transforms are often only useful when they're invertible. That is, can we find $\gamma(\omega)$ if given $f(x)$? Note that in *this* case, that's what we really care about. We are always given $f(x)$, if we can use that to find γ, then we can construct α and β in which case we have the solution to our PDE.

Dirac's Delta

We've used Kronecker's Delta extensively

$$\delta_{i,j} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

in the study of eigenfunction expansions over a discrete spectrum. A continuous version of the discrete Kronecker' delta is the Dirac delta.
Formally, Dirac's Delta is a generalized function (really, a probability distribution of a sure event) with the normalization property:

$$\int_{-\infty}^{\infty} \delta(x)\,\mathrm{d}x = 1$$

and the sifting property:

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,\mathrm{d}x = f(a).$$

(Contrast this with

$$\sum_{n=0}^{\infty} \delta_{n,a} = 1 \quad \text{and} \quad \sum_{n=0}^{\infty} c_n\,\delta_{n,a} = c_a$$

in the discrete case).