Informally, you can think of $\delta(x)$ as a limit as $n \to \infty$ of nascent-delta functions

$$\delta_n(x) = \sqrt{\frac{n}{\pi}} e^{-n\,x^2}$$

where these δ_ns are the pdf of the normal distribution with mean $\mu = 0$ and standard deviation $\sigma = \sqrt{2\,n}^{-1}$.
More formally, we can derive *one of many integral forms* of $\delta(x)$ such as

$$\delta(x) = \frac{1}{2\,\pi} \int_{-\infty}^{\infty} e^{i\,\omega\,x}\,\mathrm{d}\omega.$$

We want to derive the above form by demonstrating that it is a limit of nascent-delta functions.

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{i\,\omega\,x}\,\mathrm{d}\omega &= \int_{-\infty}^{0} e^{i\,\omega\,x}\,\mathrm{d}\omega + \int_{0}^{\infty} e^{i\,\omega\,x}\,\mathrm{d}\omega \\
&= \int_{0}^{\infty} e^{-i\,\omega\,x}\,\mathrm{d}\omega + \int_{0}^{\infty} e^{i\,\omega\,x}\,\mathrm{d}\omega \\
&= \lim_{\epsilon \to 0^+} \int_{0}^{\infty} \left(e^{-i\,\omega\,x} + e^{i\,\omega\,x}\right) e^{-\omega\,\epsilon}\,\mathrm{d}\omega \\
&= \lim_{\epsilon \to 0^+} \int_{0}^{\infty} e^{-i\,\omega\,(x - i\,\epsilon)} + e^{i\,\omega\,(x + i\,\epsilon)}\,\mathrm{d}\omega \\
&= \lim_{\epsilon \to 0^+} \left[\frac{e^{-i\,\omega\,(x - i\,\epsilon)}}{-i\,(x - i\,\epsilon)} + \frac{e^{i\,\omega\,(x + i\,\epsilon)}}{i\,(x + i\,\epsilon)}\right]_{\omega=0}^{\omega=\infty} \\
&= \lim_{\epsilon \to 0^+} \left(0 + 0 - \frac{1}{-i\,(x - i\,\epsilon)} - \frac{1}{i\,(x + i\,\epsilon)}\right) \\
&= \lim_{\epsilon \to 0^+} \frac{2\epsilon}{x^2 + \epsilon^2} = \begin{cases} 0 & x \neq 0 \\ \text{undefined} & x = 0 \end{cases}
\end{aligned}$$

Now

$$\int_{-\infty}^{\infty} \frac{1}{2\pi} \frac{2\epsilon}{x^2 + \epsilon^2}\,\mathrm{d}x = \left[\frac{1}{\pi} \arctan\left(\frac{x}{\epsilon}\right)\right]_{x=-\infty}^{x=\infty} = 1 \quad \forall \epsilon > 0$$

Armed with this we return to our separation.
Given

$$f(x) = \int_{-\infty}^{\infty} \gamma(\omega)\,e^{i\,\omega\,x}\,\mathrm{d}\omega = \int_{-\infty}^{\infty} \gamma(s)\,e^{i\,s\,x}\,\mathrm{d}s \tag{1}$$

we multiply both sides by $e^{-i\,\omega\,x}$ for some $\omega \in \mathbb{R}$

$$e^{-i\,\omega\,x}\,f(x) = \int_{-\infty}^{\infty} \gamma(s)\,e^{i\,s\,x}\,e^{-i\,\omega\,x}\,\mathrm{d}s$$

$$\begin{aligned}
\frac{1}{2\,\pi} \int_{-\infty}^{\infty} e^{-i\,\omega\,x}\,f(x)\,\mathrm{d}x &= \frac{1}{2\,\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \gamma(s)\,e^{i\,s\,x}\,e^{-i\,\omega\,x}\,\mathrm{d}s\,\mathrm{d}x \\
\frac{1}{2\,\pi} \int_{-\infty}^{\infty} e^{-i\,\omega\,x}\,f(x)\,\mathrm{d}x &= \int_{-\infty}^{\infty} \gamma(s) \left(\frac{1}{2\,\pi} \int_{-\infty}^{\infty} e^{i\,x\,(s - \omega)}\,\mathrm{d}x\right) \mathrm{d}s \\
\frac{1}{2\,\pi} \int_{-\infty}^{\infty} e^{-i\,\omega\,x}\,f(x)\,\mathrm{d}x &= \int_{-\infty}^{\infty} \gamma(s)\,\delta(s - \omega)\,\mathrm{d}s \\
\frac{1}{2\,\pi} \int_{-\infty}^{\infty} e^{-i\,\omega\,x}\,f(x)\,\mathrm{d}x &= \gamma(\omega)
\end{aligned} \tag{2}$$

Now (1) and (2) define an transformation and its inverse. However, it's "ugly" that there isn't a symmetry in the two integrals (due to the $1/(2\pi)$ term. For that reason, one typically defines $\hat{f}(\omega) = \sqrt{2\,\pi}^{-1}\,\gamma(\omega)$. In which case we get:

Definition 1 (Fourier Transform)**.** *We define the (forward) Fourier transform of an absolutely integrable function $f(x)$ as*

$$\mathcal{F}[f(x)] = \hat{f}(\omega) = \frac{1}{\sqrt{2\,\pi}}\int_{-\infty}^{\infty} f(x)\,e^{-i\,\omega\,x}\,\mathrm{d}\,x$$

and the inverse (or backward) Fourier transform as

$$\mathcal{F}^{-1}[\hat{f}(\omega)] = f(x) = \frac{1}{\sqrt{2\,\pi}}\int_{-\infty}^{\infty} \hat{f}(\omega)\,e^{i\,\omega\,x}\,\mathrm{d}\,\omega$$

Definition 2 (Absolutely Integrable)**.** *If*

$$\int_{-\infty}^{\infty} |f(x)|\,\mathrm{d}x < \infty$$

then f is absolutely integrable.

A necessary condition to being absolutely integrable is that $\lim_{x\to\pm\infty} f(x) = 0$; we'll use this later. Intuitively, the Fourier transform trades a function of x (physical space) for a function of ω (angular frequency). Hence the FT takes a function from the spatial domain to the frequency domain.
(Alternatively, we could re-scale $\omega \to 2\,\pi\,k$ in order to arrive at a "symmetric" transform/inverse transform pair, however ω represents an angular frequency and k an oscillation frequency so certain disciplines tend to use different versions of the same transform. More pure-mathematicians prefer the version in k – it doesn't require any extra constants in front of the integral; physicists and engineers tend to prefer the version in ω and differ in if they rescale constants to make the transform symmetric or not).

General Properties of the Fourier Transform

1. **Linearity** the Fourier transform is a linear operator.
2. **Space-Shift Formula** we have

$$\mathcal{F}[f(x - x_0)] = e^{-i\,\omega\,x_0}\,\hat{f}(\omega)$$

3. **Frequency Shift Formula** we have

$$\mathcal{F}[e^{i\,\omega_0\,x}\,f(x)] = \hat{f}(\omega - \omega_0)$$