4. If f is piecewise continuously differentiable, then $f(x) = \mathcal{F}^{-1}[\mathcal{F}[f(x)]]$
5. If $f(x)$ is discontinuous at x_0, then

$$\mathcal{F}^{-1}[\mathcal{F}[f(x)]](x_0) = \frac{1}{2}\left(f(x_0^-) + f(x_0^+)\right)$$

(where $\pm$ mean limits from the right/left respectively).

6. The Fourier transform preserves the value under the square norm (with unit weight function)

$$\|\hat{f}\| = \sqrt{\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 \,\mathrm{d}\omega} = \sqrt{\int_{-\infty}^{\infty} |f(x)|^2 \,\mathrm{d}x} = \|f\|$$

To see this take

$$\begin{aligned}
\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 \,\mathrm{d}\omega &= \int_{-\infty}^{\infty} \hat{f}(\omega)\,\overline{\hat{f}(\omega)}\,\mathrm{d}\omega \\
&= \frac{1}{2\,\pi}\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-i\,\omega\,x} f(x)\,\mathrm{d}x\right)\left(\int_{-\infty}^{\infty} \overline{e^{-i\,\omega\,s}\,f(s)}\,\mathrm{d}s\right)\mathrm{d}\omega \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)\,\overline{f(s)}\left(\frac{1}{2\,\pi}\int_{-\infty}^{\infty} e^{-i\,\omega\,(x-s)}\,\mathrm{d}\omega\right)\mathrm{d}s\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} f(x)\int_{-\infty}^{\infty} \overline{f(s)}\,\delta(x-s)\,\mathrm{d}s\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} |f(x)|^2\,\mathrm{d}x
\end{aligned}$$

7. Let $\hat{f}(\omega) = \mathcal{F}[f]$ and $\hat{g}(\omega) = \mathcal{F}[g]$, then

$$\begin{aligned}
\mathcal{F}^{-1}[\hat{f}(\omega)\,\hat{g}(\omega)] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\,\omega\,x}\,\hat{f}(\omega)\,\hat{g}(\omega)\,\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\,\omega\,x}\int_{-\infty}^{\infty} e^{-i\,\omega\,s} f(s)\,\mathrm{d}s\,\hat{g}(\omega)\,\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)\int_{-\infty}^{\infty} e^{i\,\omega\,(x-s)}\,\hat{g}(\omega)\,\mathrm{d}\omega\,\mathrm{d}s \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(s)\,g(x-s)\,\mathrm{d}s \\
&= \frac{1}{\sqrt{2\pi}}\,(f*g)(x)
\end{aligned}$$

8. The Fourier transform of a derivative simplifies via IBP (with $u = e^{-i\,\omega\,x}$ and $\mathrm{d}v = f'(x)\,\mathrm{d}x$)

$$\begin{aligned}
\mathcal{F}\left[\frac{\mathrm{d}f}{\mathrm{d}x}\right] &= \frac{1}{\sqrt{2\,\pi}}\int_{-\infty}^{\infty} e^{-i\,\omega\,x}\,\frac{\mathrm{d}f}{\mathrm{d}x}\mathrm{d}x \\
&= \frac{1}{\sqrt{2\,\pi}}\left(\left[e^{-i\,\omega\,x}\,f(x)\right]_{-\infty}^{\infty} + i\,\omega\int_{-\infty}^{\infty} e^{-i\,\omega\,x}\,f(x)\,\mathrm{d}x\right) \\
&= i\omega\,\mathcal{F}[f]
\end{aligned}$$

Hence, by induction, we have that

$$\mathcal{F}\left[\frac{\mathrm{d}^n f}{\mathrm{d}x^n}\right] = (i\,\omega)^n\,\mathcal{F}[f]$$

for $n \geq 1$.

It's these last properties that make Fourier transforms so useful. So-far we've only considered the Fourier transform of a function of one variable. For multivariate functions you can Fourier transform either variable. That is, you can consider the Fourier transform in space

$$\hat{u}(\omega, t) = \mathcal{F}[u(x,t)] = \int_{-\infty}^{\infty} u(x,t)\, e^{-i\,\omega\, x}\,\mathrm{d}x$$

or (on an infinite temporal domain) the Fourier transform in time

$$\hat{u}(x, \omega) = \mathcal{F}[u(x,t)] = \int_{-\infty}^{\infty} u(x,t)\, e^{-i\,\omega\, t}\,\mathrm{d}t$$

(though the former is the most common).

Application to PDEs

Example: Use the Fourier Transform method to find the general solution to the equation

$$u_{tt} + 2\gamma u_t + \gamma^2 u = c^2 u_{xx}$$

where γ and c are constants. This is a special case of the telegraph equation, which you can see reduces to the wave equation when $\gamma = 0$.

Solution: We use the Fourier transform with respect to x. We have

$$\mathcal{F}[u(x,t)] = \hat{u}(\omega, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t)\, e^{-i\omega x}\, dx$$

The inverse Fourier transform is,

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(\omega, t)\, e^{i\omega x}\, d\omega$$

Thus we have,

$$\mathcal{F}[u_{xx}(x,t)] = (i\omega)^2\, \hat{u}(\omega,t), \qquad \mathcal{F}[u_t(x,t)] = \hat{u}_t(\omega,t), \qquad \mathcal{F}[u_{tt}(x,t)] = \hat{u}_{tt}(\omega,t)$$

Thus the PDE becomes

$$\hat{u}_{tt} + 2\gamma \hat{u}_t + \gamma^2 \hat{u} = -c^2\omega^2 \hat{u}.$$

That is,

$$\hat{u}_{tt} + 2\gamma \hat{u}_t + \left(\gamma^2 + c^2\omega^2\right) \hat{u} = 0.$$

We can treat this as a second-order ODE. The characteristic equation is

$$r^2 + 2\gamma r + \left(\gamma^2 + c^2\omega^2\right) = 0$$

Thus, we have

$$r = \gamma \pm \sqrt{\gamma^2 - (\gamma^2 + c^2\,\omega^2)} = -\gamma \pm \sqrt{-c^2\,\omega^2} = -\gamma \pm i\,c\,\omega$$

Therefore, the solution is

$$\begin{aligned} \hat{u}\,(\omega, t) &= \hat{F}\,(\omega)\, e^{(-\gamma + c\,\omega\, i)\,t} + \hat{G}\,(\omega)\, e^{(-\gamma - c\,\omega\, i)\,t} \\ &= e^{-\gamma t}\left[\hat{F}\,(\omega)\, e^{c\,\omega\, i\, t} + \hat{G}\,(\omega)\, e^{-c\,\omega\, i\, t}\right]. \end{aligned}$$

Applying the inverse Fourier transform, and using the space-shift formula, we obtain

$$u\,(x, t) = e^{-\gamma t}\left[F\,(x + c\,t) + G\,(x - c\,t)\right].$$

This represents the motion of waves moving in opposite directions of exponentially decaying amplitude.

Heat Kernel

Consider the Heat equation

$$u_t = D\, u_{xx}$$

on an infinite domain with $D > 0$ for $t > 0$ with IC $u(x,0) = f(x)$. We will compute the Fourier transform in x of this IC and PDE to see

$$\hat{u}(\omega, 0) = \mathcal{F}[u(x,0)] = \mathcal{F}[f(x)] = \hat{f}(\omega)$$

and, for the PDE,

$$\begin{aligned}
\mathcal{F}[u_t] &= \mathcal{F}[D\, u_{xx}] \\
\frac{\partial}{\partial t}\mathcal{F}[u] &= D\, \mathcal{F}[u_{xx}] \\
\frac{\partial}{\partial t}\hat{u}(\omega, t) &= D\,(i\,\omega)^2\,\hat{u}(\omega) \\
\frac{\partial}{\partial t}\hat{u}(\omega, t) &= -D\,\omega^2\,\hat{u}(\omega)
\end{aligned}$$

which is an ODE that is easily solvable as

$$\hat{u}(\omega, t) = \hat{f}(\omega)\, e^{-D\,\omega^2\, t}.$$

Now this is the solution in frequency-space. To recover the solution in physical-space we must invert the transform to see

$$u(x,t) = \mathcal{F}^{-1}[\hat{f}(\omega) e^{-D\,\omega^2\, t}] = \int_{-\infty}^{\infty} f(s)\, G(x-s, t)\, \mathrm{d}s$$

by the convolution property where

$$G(x,t) = \frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}[e^{-D\,\omega^2\, t}] = \frac{1}{2\,\pi}\int_{-\infty}^{\infty} e^{i\,\omega\, x}\, e^{-D\,\omega^2\, t}\, \mathrm{d}\omega.$$

This $G(x,t)$ function is very important and is called the heat-kernel. Let's fix t and focus on the integral

$$\begin{aligned}
I(x) &= \int_{-\infty}^{\infty} e^{i\,\omega\, x} e^{-D\,\omega^2\, t}\, \mathrm{d}\omega \\
&= \int_{-\infty}^{\infty} (\cos(\omega\, x) + i\, \sin(\omega\, x)) e^{-D\,\omega^2\, t}\, \mathrm{d}\omega
\end{aligned}$$

since sin is an odd function, the imaginary part of this integral is identically zero, thus

$$I(x) = \int_{-\infty}^{\infty} \cos(\omega\, x)\, e^{-D\,\omega^2\, t}\, \mathrm{d}\omega = 2\int_{0}^{\infty} \cos(\omega\, x)\, e^{-D\,\omega^2\, t}\, \mathrm{d}\omega$$

since cos is even.
Now

$$I(0) = 2\int_{0}^{\infty} e^{-D\,\omega^2\, t}\, \mathrm{d}\omega = \sqrt{\frac{\pi}{D\,t}}$$

moreover

$$\begin{aligned}
I'(x) &= -2\int_{0}^{\infty} \omega\, \sin(\omega\, x)\, e^{-D\,\omega^2\, t}\, \mathrm{d}\omega \\
&= \frac{1}{D\,t}\left[e^{-D\,\omega^2\, t}\, \sin(\omega\, x)\right]_{\omega=0}^{\omega=\infty} - \frac{x}{D\,t}\int_{0}^{\infty} e^{-D\,\omega^2\, t}\, \cos(\omega\, x)\, \mathrm{d}\omega \\
&= -\frac{x}{2\,D\,t}\, I(x)
\end{aligned}$$

(by IBP with $u = \sin(\omega\, x)$ and $\mathrm{d}v = -2\,\omega\, e^{-D\,\omega^2\,t}\,\mathrm{d}x$). Hence we have an ODE for I complete with IC. We can solve this to see

$$I(x) = \sqrt{\frac{\pi}{D\,t}}\,\exp\left(\frac{-x^2}{4\,D\,t}\right)$$

and the heat kernel

$$G(x,t) = \frac{1}{\sqrt{4\,\pi\,D\,t}}\,\exp\left[\frac{-x^2}{4\,D\,t}\right].$$

All told, our integral solution on an infinite spatial domain is

$$u(x,t) = \frac{1}{\sqrt{4\,\pi\,D\,t}}\int_{-\infty}^{\infty}\exp\left[\frac{-(x-s)^2}{4\,D\,t}\right]\,f(s)\,\mathrm{d}s.$$

It should be clear that given the heat kernel $G(x,t)$ we can calculate the solution of the heat equation for *any* initial condition $f(x)$ via convolution

$$u(x,t) = (f * G)(x,t).$$

Now $f(x)$ is independent of t so the temporal component of $(f * G)(x,t)$ is determined *entirely* by $G(x,t)$. So for any IC of the heat equation, the solution's temporal component is determined by G. There is nothing "special" about the heat equation for determining a kernel, you can find a kernel (or *fundamental solution*) of different PDEs too. The heat equation *is* special in that the heat-kernel has a nice, closed form analytical solution.